Recitation – Week 3

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Announcements

- Office Hours: Thursday, 1-2 PM, CDS Room 650
- HW 3 due: September 24 2019

Kernel, rank and invertibility

1. Given a matrix $L \in \mathbb{R}^{m \times n}$, show that $Ker(L) = Ker(L^T L)$
2. Given a matrix $L \in \mathbb{R}^{m \times n}$, prove that $rank(L) = rank(L^T)$
3. Let $A \in \mathbb{R}^{n \times n}$ and $\mathrm{B} \in \mathbb{R}^{n \times n}$. Show that AB is invertible if and only if A and B are invertible
4. Let $A \in \mathbb{R}^{m \times n}$ where $m < n$. Under what conditions is there a matrix B such that $AB = I$ where I is the $m \times m$ identity matrix
5. Let $A \in \mathbb{R}^{m \times n}$ where $m > n$. Under what conditions is there a matrix B such that $BA = I$ where I is the $n \times n$ identity matrix

Matrix multiplication (3rd way)

Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \dots, a_m^T$ and let $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} - \ a_1^T \ - \\ - \ a_2^T \ - \\ \cdot \\ \cdot \\ \cdot \\ - \ a_m^T \ - \end{bmatrix} x = \begin{bmatrix} \ \end{bmatrix}$$

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Matrix multiplication (3rd way)

$$AX = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ x & y \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_1^T x & a_1^T y \\ a_2^T x & a_2^T y \\ \vdots & \vdots \\ a_m^T x & a_m^T y \end{bmatrix}$$

Matrix multiplication (3rd way)

$$AX = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ x & y \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_1^T x & a_1^T y \\ a_2^T x & a_2^T y \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_m^T x & a_m^T y \end{bmatrix}$$

Matrix multiplication (4th way)

Let $A \in \mathbb{R}^{m\times n}$ and B $\in \mathbb{R}^{n\times k}$. Let $a_1, \ldots, a_n$ be the columns of A and $b_1^T, \ldots, b_n^T$ denote the rows of B

$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_n^T & \cdots \end{bmatrix} =$$

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$$AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_1^T & \cdots \\ \cdots & b_2^T & \cdots \\ & \vdots & \\ \cdots & b_n^T & \cdots \end{bmatrix} = a_1 b_1^T + a_2 b_2^T +$$

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Let $A \in \mathbb{R}^{m \times n}$ and B $\in \mathbb{R}^{n \times k}$. Let $a_1, \ldots, a_n$ be the columns of A and $b_1^T, \ldots, b_n^T$ denote the rows of B

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Matrix multiplication

1. Let $x, y \in \mathbb{R}^n$ be column vectors. What's the shape of xy^T? What is it's rank?
2. Let $x, y \in \mathbb{R}^n$ be column vectors. What's the shape of $y^T x$? What is it's rank?
3. True or False: Let $A \in \mathbb{R}^{3\times2}$ and $B \in \mathbb{R}^{2\times3}$, then the rank of AB can be 3
4. Let $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$, then show that the matrix product AB can be expressed as: $AB = C_1 + \cdots + C_k$ such that $rank(C_i) \leq 1\ \forall i \in [1, k]$