Recitation Solutions - Week 6

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Oct 9th , 2019

Stochastic matrices

1. Prove that the product of two stochastic matrices is a stochastic matrix
2. Let $A \in R^{n \times \mathrm{n}}$ be a stochastic matrix. True or False:
    1. A is always invertible
    2. The eigenvector corresponding to the largest eigenvalue of A is unique
    3. A cannot have zero as its eigenvalue
3. Let $A \in \mathbb{R}^{n \times \mathrm{n}}$ be a stochastic matrix. Prove that the absolute value of any eigenvalue of A is ≤ 1

Stochastic matrices

1. Prove that the product of two stochastic matrices is a stochastic matrix

Solution: Let P_1 and P_2 be the two stochastic matrices

1. $P_1P_2[i,j] = P_1[i,:]P_2[:,j] \geq 0$
2. $[1,\dots,1]P_1P_2 = [1,\dots,1]P_2 = [1,\dots,1]$ proving that sum of entries in all columns of P_1P_2 is 1

Stochastic matrices

2. Let $A \in R^{n\times \mathrm{n}}$ be a stochastic matrix. True or False:

   1. A is always invertible
   2. The eigenvector corresponding to the largest eigenvalue of A is unique
   3. A cannot have zero as its eigenvalue

Solution:

i. False. $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

ii. False, A=Identity matrix

iii. False, $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Stochastic matrices

3. Let $A \in \mathbb{R}^{n\times \mathrm{n}}$ be a stochastic matrix. Prove that the absolute value of any eigenvalue of A is ≤ 1

Solution: For any x such that $Ax = \lambda x$

$$||Ax||_1 = \sum_i \left|\sum_j A_{ij}\, x_j\right| \leq \sum_i \sum_j |A_{ij} x_j|$$

A is a stochastic matrix

$$\Rightarrow ||Ax||_1 \leq \sum_i \sum_j A_{ij} |x_j| = \sum_j |\, x_j| \sum_i A_{ij} = \sum_j |x_j| = ||x||_1$$

$$\Rightarrow |\lambda| ||x||_1 \leq ||x||_1$$

$$\Rightarrow |\lambda| \leq 1$$

Spectral decomposition

1. Let $A \in \mathbb{R}^{n\times\mathrm{n}}$ be a symmetric matrix. Give a vector v with $||v|| = 1$ such that $||Av||$ is maximized
2. A symmetric matrix $M \in \mathbb{R}^{n\times\mathrm{n}}$ is said to be positive definite if all of its eigenvalues are greater than zero. Prove that $x^T Mx > 0$ for a symmetric positive definite matrix $M \in \mathbb{R}^{n\times\mathrm{n}}$ and $x \in \mathbb{R}^n$
3. Suppose S and T are symmetric and positive definite (all eigenvalues greater than zero)
   1. True or False: ST will always be symmetric
   2. Prove that all eigenvalues of ST are still positive

Spectral decomposition

1. Let $A \in \mathbb{R}^{n\times \mathrm{n}}$ be a symmetric matrix. Give a vector v with $||v|| = 1$ such that $||Av||$ is maximized

Solution: By spectral theorem, I can write A as $A = UDU^T$ where the columns of U (namely $u_1, \dots, u_n$) form an orthonormal basis of A and let the diagonal entries in D be $d_1, .., d_n$

Let $\mathrm{v} = \alpha_1 u_1 + \cdots + \alpha_n u_n$ such that $\sqrt{\alpha_1^2 + \cdots \alpha_n^2} = 1$

$$Av = UDU^T v = \alpha_1 d_1 u_1 + \cdots + \alpha_n d_n u_n$$

$$\Rightarrow ||Av|| = \sqrt{\alpha_1^2 d_1^2 + \cdots + \alpha_n d_n^2}$$

Let $d_k^2 = \max\{d_1^2, \dots, d_n^2\}$

$$\Rightarrow ||Av|| = |d_k| \sqrt{\alpha_1^2 \left(\frac{d_1}{d_k}\right)^2 + \cdots + \alpha_n^2 \left(\frac{d_n}{d_k}\right)^2} \leq |d_k| \sqrt{\alpha_1^2 + \cdots + \alpha_n^2} = |d_k|$$

$\Rightarrow ||Av|| \leq |d_k|$ and the maximum is achieved for $v = u_k$

Spectral decomposition

2. A symmetric matrix $A \in \mathbb{R}^{n\times \mathrm{n}}$ is said to be positive definite if all of its eigenvalues are greater than zero. Prove that $x^T Ax > 0$ for a symmetric positive definite matrix $A \in \mathbb{R}^{n\times \mathrm{n}}$ and $x \in \mathbb{R}^n$

Solution: Following the notation from previous question,

$$Ax = \alpha_1 d_1 u_1 + \cdots + \alpha_n d_n u_n$$

$$\Rightarrow x^T Ax = \alpha_1^2 d_1 + \cdots + \alpha_n^2 d_n > 0 \; since \;\; d_1, \ldots, d_n > 0$$

Spectral decomposition

3. Suppose S and T are symmetric and positive definite (all eigenvalues greater than zero)

   1. True or False: Product of two symmetric matrices is always symmetric
   2. Prove that all eigenvalues of ST are still positive

Solution:

1. $S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \Rightarrow S_1 S_2 = \begin{bmatrix} 7 & 7 \\ 5 & 8 \end{bmatrix}$ (Not symmetric)
2. Let x be an eigenvector of ST corresponding to eigenvalue $\lambda \Rightarrow STx = \lambda x$

$$\Rightarrow (Tx)^T\ STx = \lambda (Tx)^T x$$

$$\Rightarrow x^T T^T STx = \lambda x^T Tx$$

$\Rightarrow \lambda = v^T Sv / x^T Tx$ where $v = Tx$

Since S and T are positive definite, $x^T Tx > 0\ and\ v^T Sv > 0$

$$\Rightarrow \lambda > 0$$