Recitation - Week 5

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Eigenvalues and diagonalizable matrices

1. Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue λ with eigenvector $v \in \mathbb{R}^n$ associated to λ. Prove that
   i. $\forall\, \alpha \in \mathbb{R}, \lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha I_{n \times \mathrm{n}}$ with corresponding eigenvector v
   ii. $\forall\, k \in \mathbb{N}, \lambda^k$ is an eigenvalue of the matrix A^k with corresponding eigenvector v
2. Let $A \in \mathbb{R}^{m \times n}$, $\mathrm{B} \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{n \times n}$. Prove that if B and C are invertible, then rank(A)=rank(BAC). What does this say about rank of any diagonalizable matrix $X \in \mathbb{R}^{n \times n}$? Prove that $Tr(X) = \sum_i \lambda_i$ where λ_i are the eigenvalues of X.
3. Let $v_1, \dots, v_k$ be the eigenvectors of A corresponding respectively to eigenvalues $\lambda_1, \dots, \lambda_k$ such that all λ_i are distinct (assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_k$). Prove the eigenvectors $v_1, \dots, v_k$ are linearly independent

Eigenvalues and diagonalizable matrices

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   i. $\forall\, \alpha \in \mathbb{R}, \lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha I_{n\times \mathrm{n}}$ with corresponding eigenvector v
   ii. $\forall\, k \in \mathbb{N}, \lambda^k$ is an eigenvalue of the matrix A^k with corresponding eigenvector v

Solution:

i. $(A + \alpha I)v = Av + \alpha Iv = \lambda v + \alpha v = (\lambda + \alpha)v$

ii. $A^k v = A.A \dots .A(v)$ (A multiplied k times)

$\Rightarrow A^k v = A.A \dots A(Av) = A \dots .A(\lambda v) = \lambda A^{k-1}v = \lambda A^{k-2}Av = \lambda^2 A^{k-2} v = \cdots = \lambda^k v$

Eigenvalues and diagonalizable matrices

2. Let $A \in \mathbb{R}^{m\times n}$, B $\in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times n}$. Prove that if B and C are invertible, then rank(A)=rank(BAC). What does this say about rank of any diagonalizable matrix $X \in \mathbb{R}^{n\times n}$? Prove that $Tr(X) = \sum_i \lambda_i$ where λ_i are the eigenvalues of X

Solution:

From HW3, we know that if M is an invertible matrix, then rank(MA)=rank(A) and rank(AM)=A. Since B and C are invertible matrices, we can use these 2 results on BAC

$$\Rightarrow Rank(BAC) = Rank\big(B(AC)\big) = rank(AC) = rank(A)$$

Since X is an invertible matrix,

$$X = PDP^{-1}$$

$$\Rightarrow rank(X) = rank(D)$$

Also, from HW3, we know that Tr(AB)=Tr(BA)

$$\Rightarrow Tr(X) = Tr(PDP^{-1}) = Tr(P^{-1}PD) = Tr(D) = \sum_i \lambda_i$$

Eigenvalues and diagonalizable matrices

3. Let $v_1, \dots, v_k$ be the eigenvectors of A corresponding respectively to eigenvalues $\lambda_1, \dots, \lambda_k$ such that all λ_i are distinct (assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_k$). Prove the eigenvectors $v_1, \dots, v_k$ are linearly independent

Solution:

Let's take matrix $B = A + |\lambda_k| I$. We know that the eigenvalues of matrix B are the eigenvalues of A shifted by $|\lambda_k|$

$\Rightarrow \gamma_i = \lambda_i + |\lambda_k|$ are the eigenvalues of matrix B for $i \in [1, k]$. We do this so that all eigenvalues are non-negative.

To check that $v_1, \dots, v_k$ are linearly independent,

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$$

$$\Rightarrow B^p(\alpha_1 v_1 + \cdots + \alpha_k v_k) = 0$$

$$\Rightarrow \alpha_1 B^p v_1 + \cdots + \alpha_k B^P v_k = 0$$

$$\Rightarrow \gamma_1^p \left(\alpha_1 v_1 + \alpha_2 \left(\frac{\gamma_2}{\gamma_1}\right)^p v_2 + \cdots + \alpha_k \left(\frac{\gamma_k}{\gamma_1}\right)^p v_k\right) = 0$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 \left(\frac{\gamma_2}{\gamma_1}\right)^p v_2 + \cdots + \alpha_k \left(\frac{\gamma_k}{\gamma_1}\right)^p v_k = 0$$

Eigenvalues and diagonalizable matrices

$$\frac{\gamma_i}{\gamma_1} < 1 \; \forall \; i \in [2, k]$$

Taking $\lim_{p\to\infty} \left(\alpha_1 v_1 + \alpha_2 \left(\frac{\gamma_2}{\gamma_1}\right)^p v_2 + \cdots + \alpha_k \left(\frac{\gamma_k}{\gamma_1}\right)^p v_k \right) = 0$

$$\Rightarrow \alpha_1 v_1 = 0$$

Now since eigenvector cannot be a zero vector, $\alpha_1 = 0$

Now we are left with $\alpha_2 v_2 + \cdots + \alpha_k v_k = 0$. We can multiply this with B^p and do the same exact procedure as above but this time, we divide the equation with γ_2^p instead of γ_1^p. This results in $\alpha_2 = 0$. Keep on doing this and at the end you'll be left with $\alpha_k v_k = 0$ which directly gives $\alpha_k = 0$. Thus, now you have $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ possible only if $\alpha_1 = \cdots = \alpha_k = 0$

Gram-Schmidt

1. Let $A \in \mathbb{R}^{m \times \mathrm{n}}$ have linearly independent columns. Show that there is a matrix $Q \in \mathbb{R}^{m \times \mathrm{n}}$ and $R \in \mathbb{R}^{n \times \mathrm{n}}$ such that A=QR, Q has orthonormal columns and R is upper triangular
2. Suppose $D \in \mathbb{R}^{n \times n}$ is diagonal. Give a vector $v \in \mathbb{R}^n$ with $||v|| = 1$ such that $||Dv||$ is maximized

Gram-Schmidt

1. Let $A \in \mathbb{R}^{m\times \mathrm{n}}$ have linearly independent columns. Show that there is a matrix $Q \in \mathbb{R}^{m\times \mathrm{n}}$ $and\ R \in \mathbb{R}^{n\times \mathrm{n}}$ such that A=QR, Q has orthonormal columns and R is upper triangular

Solution:

Let the columns of A be $a_1, \dots, a_n$. We pass them through Gram-Schmidt and get as output a set of orthonormal vectors $u_1, \dots, u_n$. Because we used Gram-Schmidt we know that $span(a_1, \dots, a_i) = span(u_1, .., u_i) \Rightarrow a_i \in span(u_1, \dots, u_i)\ \forall\ i \in [1, n]$

$$\Rightarrow a_i = \lambda_{1i} u_1 + \cdots + \lambda_{ii} u_i$$

Now since A is a matrix of $a_1, \dots, a_n$, I can construct the matrix Q such that the columns of Q are $u_1, \dots, u_n$. The matrix R is then an upper triangular matrix constructed using the coefficients λ_{ij}

Gram-Schmidt

2. Suppose $D \in \mathbb{R}^{n\times n}$ is diagonal. Give a vector $v \in \mathbb{R}^n$ with $||v|| = 1$ such that $||Dv||$ is maximized

Solution:

Let the diagonal of D consist of $d_1, d_2, \dots, d_n$ and $v = (v_1, \dots, v_n)$. Then

$$||Dv|| = \sqrt{d_1^2 v_1^2 + \cdots + d_n^2 v_n^2} = \sqrt{d_i^2 [\left(\frac{d_1}{d_i}\right)^2 v_1^2 + .. + \left(\frac{d_n}{d_i}\right)^2 v_n^2]} \text{ where } d_i^2 = \max(d_1^2, \dots, d_n^2)$$

$$\Rightarrow ||Dv|| = \sqrt{d_i^2 [\left(\frac{d_1}{d_i}\right)^2 v_1^2 + .. + \left(\frac{d_n}{d_i}\right)^2 v_n^2]} = |d_i| \sqrt{\left(\frac{d_1}{d_i}\right)^2 v_1^2 + .. + \left(\frac{d_n}{d_i}\right)^2 v_n^2} \leq |d_i| \sqrt{v_1^2 + \cdots + v_n^2}$$

Since $||v|| = 1$,

$$||Dv|| \leq |d_i|$$

Now the question is if this maximum is achievable

I can take v as vector of all zeros but 1 at the i^{th} position to see that $||Dv|| = |d_i|$, proving that this maxima is indeed possible to get to