Recitation - Week 4

Ashwin Bhola

CDS, NYU

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Projections

1. Suppose $v_1, .., v_m \in \mathbb{R}^k$ are m non-zero orthogonal vectors. Prove that they are linearly independent.
2. Let U be the subspace of $\mathbb{R}^n$ with orthonormal basis $u_1, \ldots, u_k$.
   i. Prove that the projection of $x \in \mathbb{R}^n$ can be expressed as $P_U(x) = < u_1, x > u_1 + < u_2, x > u_2 + \ldots + < u_k, x > u_k$ (Use the fact that the orthonormal basis for a subspace of $\mathbb{R}^n$ can be extended to obtain an orthonormal basis for $\mathbb{R}^n$)
   ii. Prove that $||P_U(x)|| \leq ||x||$
   iii. Prove that $x - P_U(x)$ is orthogonal to the subspace U
   iv. Show that the linear transformation $P_U: \mathbb{R}^n \to \mathbb{R}^n$ satisfies $P_U^2 = P_U \; and \; P_U^T = P_U$

Projections

1. Suppose $v_1, .., v_m \in \mathbb{R}^k$ are m non-zero orthogonal vectors. Prove that they are linearly independent.

Solution:

To check if they are linearly independent, we need to calculate $\alpha \in \mathbb{R}^m$ such that

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$$

For any index $i \in [1, m]$,

$$< \alpha_1 v_1 + .. + \alpha_m v_m, v_i > = < 0, v_i > = 0$$

$$\Rightarrow \alpha_1 < v_1, v_i > + \cdots + \alpha_i < v_i, v_i > + \cdots + \alpha_m < v_m, v_i \geq 0$$

$$\Rightarrow \alpha_i < v_i, v_i > = 0$$

Since $v_i \neq 0, < v_i, v_i > \neq 0 \Rightarrow \alpha_i = 0$ for all $i \in [1, m]$

Projections

2. Let U be the subspace of $\mathbb{R}^n$ with orthonormal basis $u_1, \dots, u_k$.

   i. Prove that the projection of $x \in \mathbb{R}^n$ can be expressed as $P_U(x) =< u_1, x > u_1 +< u_2, x > u_2 +....+< u_k, x > u_k$ (Use the fact that the orthonormal basis for a subspace of $\mathbb{R}^n$ can be extended to obtain an orthonormal basis for $\mathbb{R}^n$)

   ii. Prove that $||P_U(x)|| \leq ||x||$

   iii. Prove that $x - P_U(x)$ is orthogonal to the subspace U

   iv. Show that the linear transformation $P_U: \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $P_U^2 = P_U \ and \ P_U^T = P_U$

Projections

Solution: The orthonormal basis of U is $u_1, \dots, u_k$. This can be extended to form a basis $u_1, \dots, u_n$ of $\mathbb{R}^n$

$i.$ $x = \alpha_1 u_1 + \cdots + \alpha_k u_k$ where $\alpha_i = < u_i, x > \forall\, i \in [1, n]$

For projection, we need to solve

$$P_U(x) = argmin_{y \in U} ||x - y||$$

Since $y \in U$,

$$y = \beta_1 u_1 + \cdots + \beta_k u_k$$

$$\big||x - y|\big| = \big||(\alpha_1 - \beta_1)u_1 + \cdots + (\alpha_k - \beta_k)u_k + \alpha_{k+1}u_{k+1} + \cdots + \alpha_n u_n|\big|$$

$$\Rightarrow \big||x - y|\big| = \sqrt{(\alpha_1 - \beta_1)^2 + \cdots + (\alpha_k - \beta_k)^2 + \alpha_{k+1}^2 + \cdots + \alpha_n^2}$$

This will be minimized when $\beta_i = \alpha_i \;\forall\, i \in [1, k]$

$$\Rightarrow P_U(x) = \alpha_1 u_1 + \cdots + \alpha_k u_k$$

$$\Rightarrow P_U(x) = < u_1, x > u_1 + \cdots + < u_k, x > u_k$$

Projections

ii. $||P_U(x)|| = \sqrt{\alpha_1^2 + \cdots + \alpha_k^2}$ and $||x|| = \sqrt{\alpha_1^2 + \cdots \alpha_k^2 + \cdots + \alpha_n^2}$ $where\ \alpha_i =< u_i, x >$

$$\Rightarrow P_U(x) \leq ||x||$$

iii. To show that a vector is orthogonal to a subspace U, it is enough to show that it is orthogonal to the Basis of U

$$x - P_U(x) = \alpha_{k+1}u_{k+1} + \cdots + \alpha_n u_n$$

For any element u_i of the basis $u_1, \dots, u_k$,

$$< u_i, x - P_U(x) > = < \mathrm{u_i}, \alpha_{k+1}u_{k+1} + \cdots + \alpha_n u_n >$$

$$= \alpha_{k+1} < u_i, u_{k+1} > + \cdots + \alpha_n < u_i, u_n >$$

$$= \alpha_{k+1}.0 + \cdots + \alpha_n.0 \text{ (given that } u_1, \dots, u_n \text{ are orthonormal)}$$

$$= 0$$

Thus, $< x - P_U(x), u_i >= 0\ \forall\ i \in [1, \dots, k]$

Projections

iv. $P_U = VV^T$ $where\ V = matrix\ with\ columns\ u_1, \dots, u_k$

$$P_U^2 = P_U P_U = VV^TVV^T$$

Now, $V^TV = \begin{bmatrix} \cdots & u_1^T & \cdots \\ & \vdots & \\ \cdots & u_k^T & \cdots \end{bmatrix} \begin{bmatrix} \vdots & & \vdots \\ u_1 & \cdots & u_k \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \mathrm{u}_1^T u_1 & \cdots & u_1^T u_n \\ \vdots & & \vdots \\ u_n^T u_1 & \cdots & u_n^T u_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = Id_{k \times k}$

$$\Rightarrow P_U^2 = VIV^T = VV^T = P_U$$

$P_U^T = (VV^T)^T = VV^T = P_U$

Norm and inner product

1. Which of the following are an inner product for $x, y \in \mathbb{R}^3$:
   i. $f(x, y) = x_1 y_2 + x_2 y_3 + x_3 y_1$
   ii. $f(x, y) = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$
   iii. $f(x, y) = x_1 y_1 + x_2 y_2$
2. Let $x = (cos\theta_1, sin\theta_1) \in \mathbb{R}^2$ and $y = (cos\theta_2, sin\theta_2) \in \mathbb{R}^2$ be two vectors on unit circle ($||x|| = 1 = ||y||$). What does $x^T y$ represent geometrically?
3. For $x, y \in \mathbb{R}^n$, prove that $||x + y|| \leq ||x|| + ||y||$ (Triangular inequality). When does $||x + y|| = ||x|| + ||y||$?
4. For any $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, show that

$$||Ax|| \leq ||x|| \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

Norm and inner product

1. Which of the following are an inner product for $x, y \in \mathbb{R}^3$:

   i. $f(x, y) = x_1y_2 + x_2y_3 + x_3y_1$

   ii. $f(x, y) = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2$

   iii. $f(x, y) = x_1y_1 + x_2y_2$

Solution:

i. Not an inner product. Take $x = y = (1, -1, 0); \mathrm{f(x, x)} = -1 < 0$

ii. Not an inner product: $f(\alpha x, y) = \alpha^2 f(x, y) \neq \alpha f(x, y)$

iii. Not an inner product: Take $x = y = (0, 0, -1) \Rightarrow f(x, x) = 0 \; and \; x \neq 0$

Norm and inner product

2. Let $x = (cos\theta_1, sin\theta_1) \in \mathbb{R}^2$ and $y = (cos\theta_2, sin\theta_2) \in \mathbb{R}^2$ be two vectors on unit circle ($||x|| = 1 = ||y||$). What does $x^T y$ represent geometrically?

Solution:

$$x^T y = cos\theta_2 cos\theta_1 + sin\theta_2 sin\theta_1$$
$$\Rightarrow x^T y = \cos(\theta_2 - \theta_1)$$

Norm and inner product

3. For $x, y \in \mathbb{R}^n$, prove that $||x+y|| \leq ||x|| + ||y||$ (Triangular inequality). When does $||x+y|| = ||x|| + ||y||$?

Solution:

$$||x+y||^2 =< x+y, x+y > =< x, x >+< y, y >+< x, y >+< y, x >$$

$$\Rightarrow ||x+y||^2 = ||x||^2 + ||y||^2 + 2 < x, y >$$

Using Cauchy-Schwarz inequality,

$$\Rightarrow ||x+y||^2 \leq ||x||^2 + ||y||^2 + 2||x||||y||$$

$$\Rightarrow ||x+y||^2 \leq \left(||x|| + ||y||\right)^2$$

Since norms are always positive,

$$||x+y|| \leq ||x|| + ||y||$$

For equality to hold, we should have equality in Cauchy-Shwarz i.e. $||x||||y|| =< x, y >$ which is true iff x and y are collinear

Norm and inner product

4. For any $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, show that

$$||Ax|| \leq ||x|| \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

Solution: Let $a_1^T, \ldots, a_m^T$ be the columns of A

$$||Ax||^2 = (a_1^T x)^2 + (a_2^T x)^2 + \cdots + (a_m^T x)^2$$

Using Cauchy-Schwarz,

$$||Ax||^2 \leq ||a_1||^2 ||x||^2 + \cdots + ||a_m||^2 ||x||^2$$

$$\Rightarrow ||Ax|| \leq \left(||a_1||^2 ||x||^2 + \cdots + ||a_m||^2 ||x||^2\right)^{0.5}$$

$$\Rightarrow ||Ax|| \leq ||x|| \left(||a_1||^2 + \cdots + ||a_m||^2\right)^{0.5}$$

Since $a_1^T, \ldots, a_m^T$ are rows of A, $||a_i||^2 = a_{i,1}^2 + \cdots + a_{i,n}^2$ assuming $a_i = [a_{i,1}, \ldots, a_{i,n}]$

$$\Rightarrow ||Ax|| \leq ||x|| \left(\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2\right)^{0.5}$$