Recitation Solutions – Week 2

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Announcements

Office hour timings: Thursday, 1-2 PM, CDS Room 650

HW 2 due: 17th Sept 2019

Linear transformations

1. Project every vector $v \in \mathbb{R}^3$ onto the plane $z = 0$. How is this transformation defined? Is this a linear transformation? If yes, what's the matrix corresponding to this transformation? Also, what is the kernel and image of this transformation?
2. Which of the following functions are linear?
   a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(v_1, v_2) = (v_2, 4v_1 + v_2, 0)$
   b) $T: \mathbb{R}^2 \to \mathbb{R}$ such that $T(v_1, v_2) = v_1 - v_2 + 5$
   c) $T: \mathbb{R}^2 \to \mathbb{R}$ such that $T(v_1, v_2) = \sqrt{\mathrm{v}_1^2 + v_2^2}$

3. Given a linear transformation $L: \mathbb{R}^m \to \mathbb{R}^n$, show that $\ker(L)$ is a subspace of $\mathbb{R}^m$ and $Im(L)$ is a subspace of $\mathbb{R}^n$

Linear transformations

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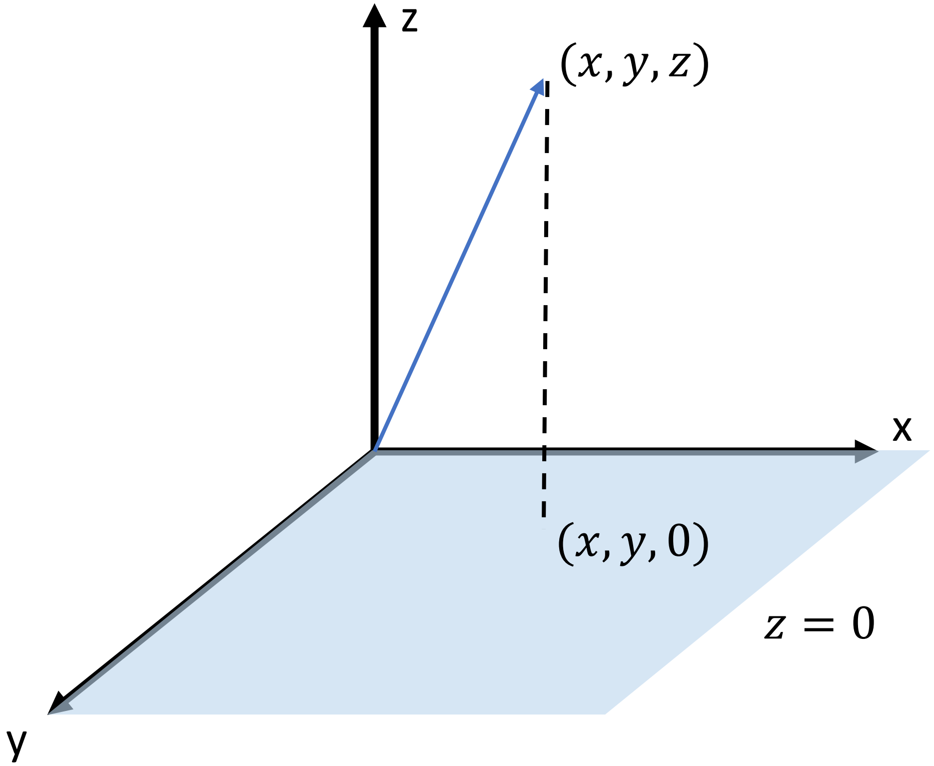

Linear transformations

1. Project every vector $v \in \mathbb{R}^3$ onto the plane $z = 0$. How is this transformation defined? Is this a linear transformation? If yes, what's the matric corresponding to this transformation? Also, what is the kernel and image of this transformation?

Solution: The transformation is defined by: $L(x, y, z) = (x, y, 0)$

$$\Rightarrow L(x_1, y_1, z_1) = (x_1, y_1, 0)\ and\ L(x_2, y_2, z_2) = (x_2, y_2, 0)$$

$$\Rightarrow L(x_1, y_1, z_1) + L(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, 0) = L(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Also,

$$L(\alpha x, \alpha y, \alpha z) = (\alpha x, \alpha y, 0) = \alpha(x, y, 0) = \alpha L(x, y, z)$$

This shows that L is a linear transformation

$$L \in \mathbb{R}^{3\times 3} = \begin{bmatrix} \vdots & \vdots & \vdots \\ L(e_1) & L(e_2) & L(e_3) \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$L(e_1) = L(1,0,0) = (1,0,0)$$
$$L(e_2) = L(0,1,0) = (0,1,0)$$
$$L(e_3) = L(0,0,1) = (0,0,0)$$

Linear transformations

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ker(L) = \{v \in \mathbb{R}^3 | Lv = 0\} = \{(x, y, z) | L(x, y, z) = 0\}$$

Now, $L(x, y, z) = (x, y, 0)$

$$\Rightarrow Ker(L) = \{(x, y, z) | (x, y, 0) = 0\}$$

$$\Rightarrow Ker(L) = \{(0,0, z) | z \in \mathbb{R}\}$$

$$Im(L) = \{Lv | v \in \mathbb{R}^3\} = \{(x, y, 0) | x, y \in \mathbb{R}\}$$

Linear transformations

Which of the following functions are linear?

a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(v_1, v_2) = (v_2, 4v_1 + v_2, 0)$

b) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $T(v_1, v_2) = v_1 - v_2+5$

c) $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $T(v_1, v_2) = \sqrt{\mathrm{v}_1^2 + v_2^2}$

Solution:

a) Linear (proof same as last question)

b) Not linear because $T(0,0) \neq 0$

c) Not linear because $T(0,1) + T(1,0) = 2 \neq T(1,1)$

Linear transformations

Given a linear transformation $L: \mathbb{R}^m \to \mathbb{R}^n$, show that $\ker(L)$ is a subspace of $\mathbb{R}^m$ and $Im(L)$ is a subspace of $\mathbb{R}^n$

Solution:

$$Ker(L) = \{x \in \mathbb{R}^m | Lx = 0\}$$

Ker(L) is a subset of $\mathbb{R}^m$. To show that it is a subspace:

i. Let $u, v \in Ker(L) \Rightarrow Lu = 0 \; and \; Lv = 0$

$$\Rightarrow Lu + Lv = 0 \Rightarrow L(u + v) = 0 \Rightarrow u + v \in Ker(L)$$

ii. Let $u \in Ker(L) \Rightarrow Lu = 0$. For any scalar $\alpha \in \mathbb{R}$,

$$\alpha Lu = \alpha . 0 = 0$$

$$\Rightarrow L(\alpha u) = 0 \Rightarrow \alpha u \in Ker(L)$$

iii. $L.0 = 0 \Rightarrow 0 \in Ker(L)$

Linear transformations

$$Im(L) = \{Lx \in \mathbb{R}^n | x \in \mathbb{R}^m\}$$

Im(L) is a subset of $\mathbb{R}^n$. To show that it is a subspace:

i. Let $u, v \in Im(L) \Rightarrow \exists x, y \in \mathbb{R}^m \; such\ that\ Lx = u\ and\ Ly = v$

$$\Rightarrow Lx + Ly = u + v \Rightarrow L(x+y) = u + v \Rightarrow u + v \in Im(L)$$

ii. Let $u \in Im(L) \Rightarrow \exists\ x \in \mathbb{R}^m \; such\ that\ Lx = u$. For any scalar $\alpha \in \mathbb{R}$,

$$\alpha Lx = \alpha . u$$

$$\Rightarrow L(\alpha x) = \alpha u \Rightarrow \alpha u \in Im(L)$$

iii. $L.0 = 0 \Rightarrow 0 \in Im(L)$

Matrix multiplication (2 new ways)

- Let $A \in \mathbb{R}^{m \times n}$ with columns $a_1, \ldots, a_n$ and let $x = [x_1, \ldots., x_n]^T \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} \vdots & \vdots & \vdots \\ a_1 & \cdots & a_n \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \vdots \\ a_1 \\ \vdots \end{bmatrix} + \cdots + x_n \begin{bmatrix} \vdots \\ a_n \\ \vdots \end{bmatrix}$$

- Let $A \in \mathbb{R}^{m \times n}$ with rows $a_1^T, \ldots, a_m^T$ and let $x = [x_1, \ldots., x_m] \in \mathbb{R}^m$

$$xA = [x_1 \ldots x_m] \begin{bmatrix} - \; a_1^T \; - \\ \vdots \\ - \; a_m^T \; - \end{bmatrix} = x_1[\cdots a_1^T \cdots] + \cdots + x_m[\cdots a_m^T \cdots]$$

Both these methods can be easily extended for cases where x is a matrix

Matrix multiplication

1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 4 \\ 5 & 2 & 1 \end{bmatrix}$
   a) How can you swap the the first and third column of A via matrix multiplication?
   b) How can you replace the second row with twice the first row added to the second row of A and then swap the obtained second row with the third row via matrix multiplication?
2. Fix $A \in \mathbb{R}^{4\times 5}$. Describe the following set:

$$\left\{ Ax : x = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ c \end{bmatrix}, a, b, c \in \mathbb{R} \right\}$$

Matrix multiplication

1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 4 \\ 5 & 2 & 1 \end{bmatrix}$
   a) How can you swap the the first and third column of A via matrix multiplication?
   b) How can you replace the second row with twice the first row added to the second row of A and then swap the obtained second row with the third row via matrix multiplication?

Solution:

a) Since we operate on columns, we'll multiply X on the right of A (method 1 of matrix multiplication)

$$AX = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 4 \\ 5 & 2 & 1 \end{bmatrix} X = \begin{bmatrix} 0 & 2 & 1 \\ 4 & 0 & 1 \\ 1 & 2 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Matrix multiplication

b) Since we operate on rows, we'll multiply X on the left of A (method 2 of matrix multiplication)

$$XA = X\begin{bmatrix} \cdots & R_1 & \cdots \\ \cdots & R_2 & \cdots \\ \cdots & R_3 & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & R_1 & \cdots \\ \cdots & R_3 & \cdots \\ \cdots & 2R_1 + R_2 & \cdots \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Linear transformations and revisiting basis

1. If $L: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation such:

$$L(1,2) = (1,3,0)$$
$$L(2,3) = (0,1,1)$$

   Write the matrix representation of L.

2. Prove that any basis of $\mathbb{R}^n$ has length n

Linear transformations and revisiting basis

1. If $L: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such:

$$L(1,2) = (1,3,0)$$
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Write the matrix representation of L.

Solution:

$$Let\ L = \begin{bmatrix} \vdots & \vdots \\ L_1 & L_2 \\ \vdots & \vdots \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

$$L(1,2) = (1,3,0) \Rightarrow L_1 + 2L_2 = (1,3,0)$$
$$L(2,3) = (0,1,1) \Rightarrow 2L_1 + 3L_2 = (0,1,1)$$

Now we have 2 equations in 2 variables. Solving them,

$$L_1 = (-3,-7,2)\ and\ L_2 = (2,5,-1)$$

$$\Rightarrow L = \begin{bmatrix} -3 & 2 \\ -7 & 5 \\ 2 & -1 \end{bmatrix}$$

2. Prove that any basis of $\mathbb{R}^n$ has length n

Solution:
From lecture notes, we know that:
Lemma 3.1: Let $v_1, .., v_m$ span $\mathbb{R}^n$ and suppose $w_1, \dots, w_p \in \mathbb{R}^n$ with $p > m$. Then $w_1, \dots, w_p$ are linearly dependent

For $\mathbb{R}^n$, we already have a canonical basis $e_1, \dots, e_n$ which is a set of n vectors in $\mathbb{R}^n$ that are linearly independent and span $\mathbb{R}^n$. Now, using lemma 3.1, this automatically implies that any set of vectors having >n number of elements is linearly dependent, and thus cannot be the basis.
On the other hand, if we can find a set of vectors with <n number of elements in $\mathbb{R}^n$ that span $\mathbb{R}^n$, then $e_1, \dots, e_n$ is linearly dependent, which we know isn't true.