Vector Spaces-Solutions

Ashwin Bhola

CDS, NYU

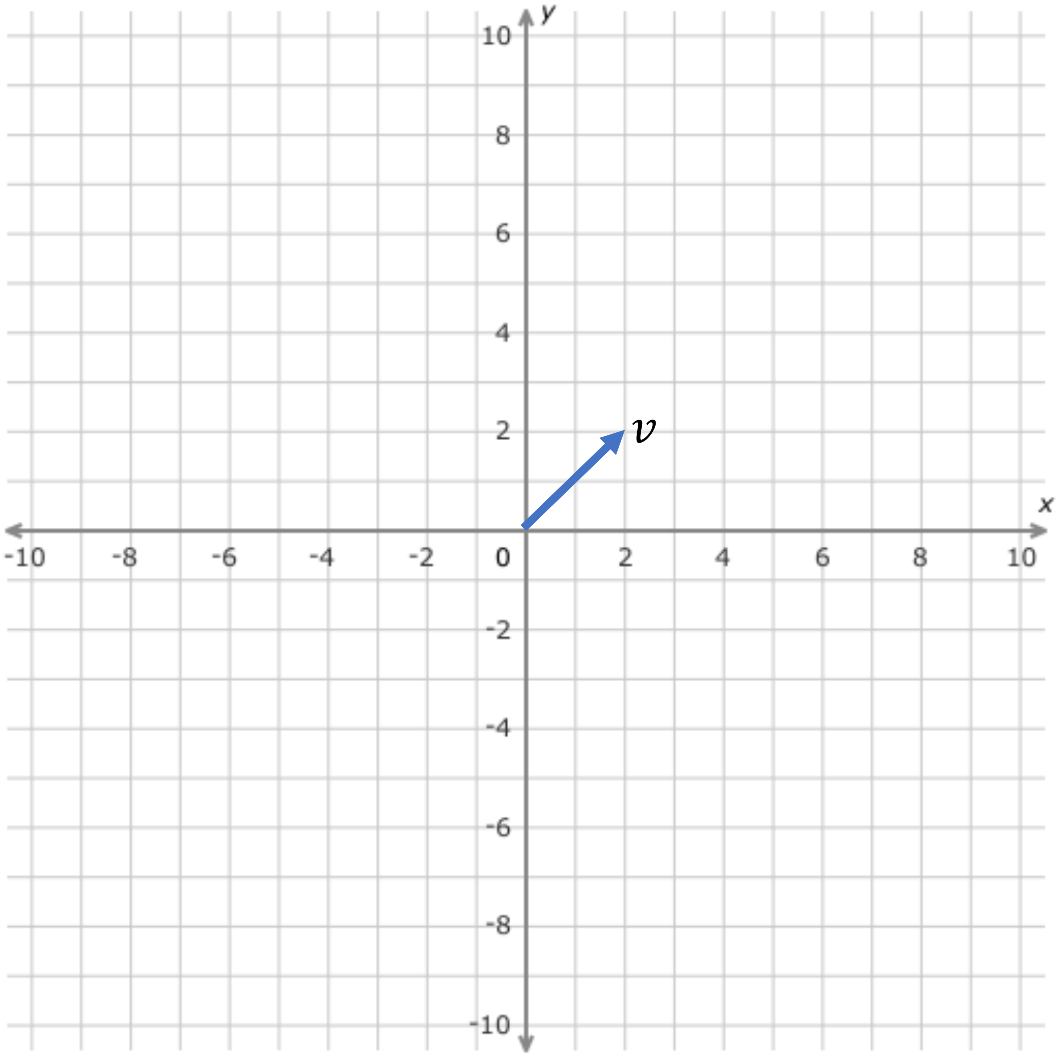
Sept 4^{th}, 2019

Visualizations in $\mathbb{R}^2$

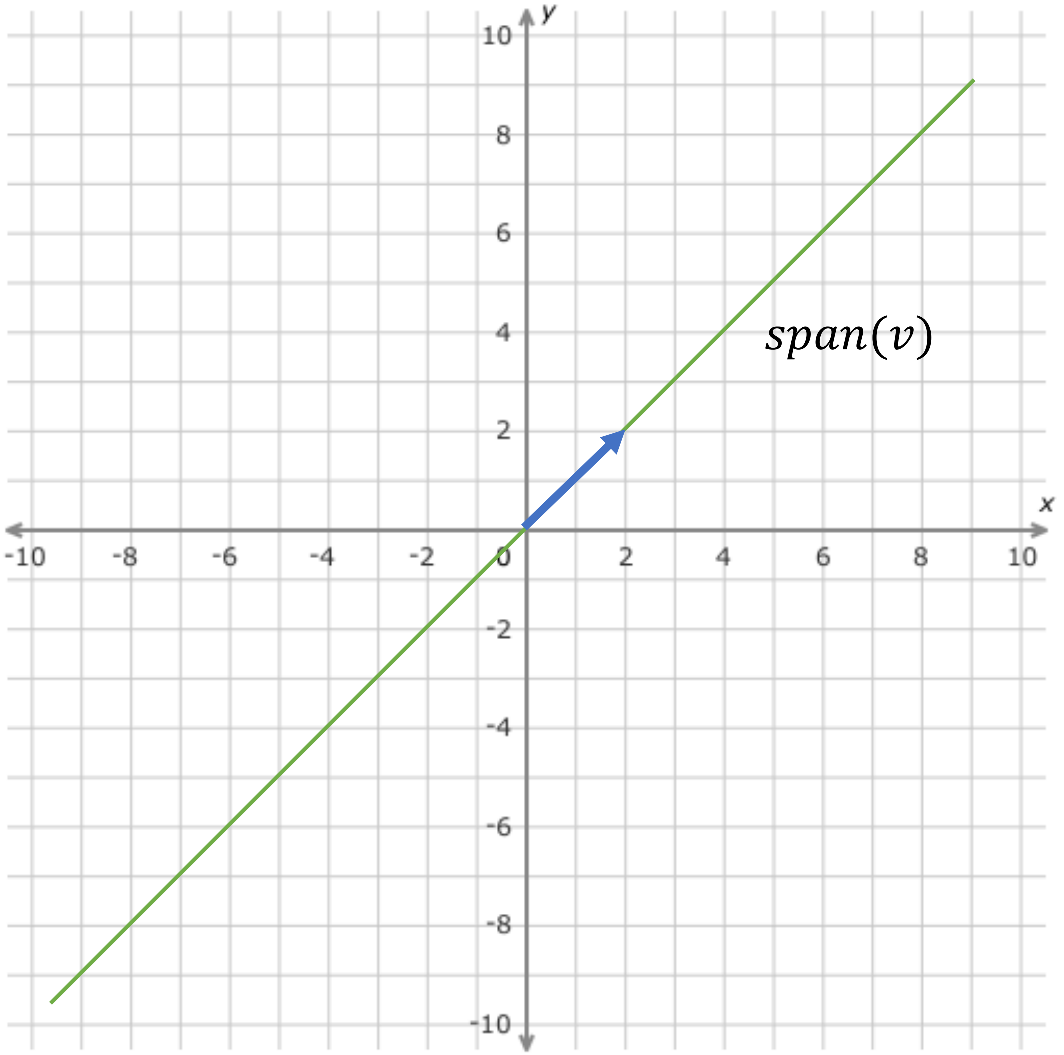
Consider 2 vectors *v* and *w* in $\mathbb{R}^2$. Let $v = (2,2)$ and $\mathrm{w} = (-2,3)$. Interpret the following sets geometrically. Which of these are a subspaces of $\mathbb{R}^2$?

- Span(*v*)
- Span(*v*) ∪ Span(*w*)
- Span(*v*) ∩ Span(*w*)
- Span(*v, w*)
- $\{(1-t)v + tw : t \in (0,1)\}$
- $\{(1-t)v + tw : t \in \mathbb{R}\}$
- $\{av + bw : a, b \geq 0\}$
- $\{(a, b) \in \mathbb{R}^2 : \mathrm{a}^2 + \mathrm{b}^2 \leq 25\}$
- $\{(a, a+5) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

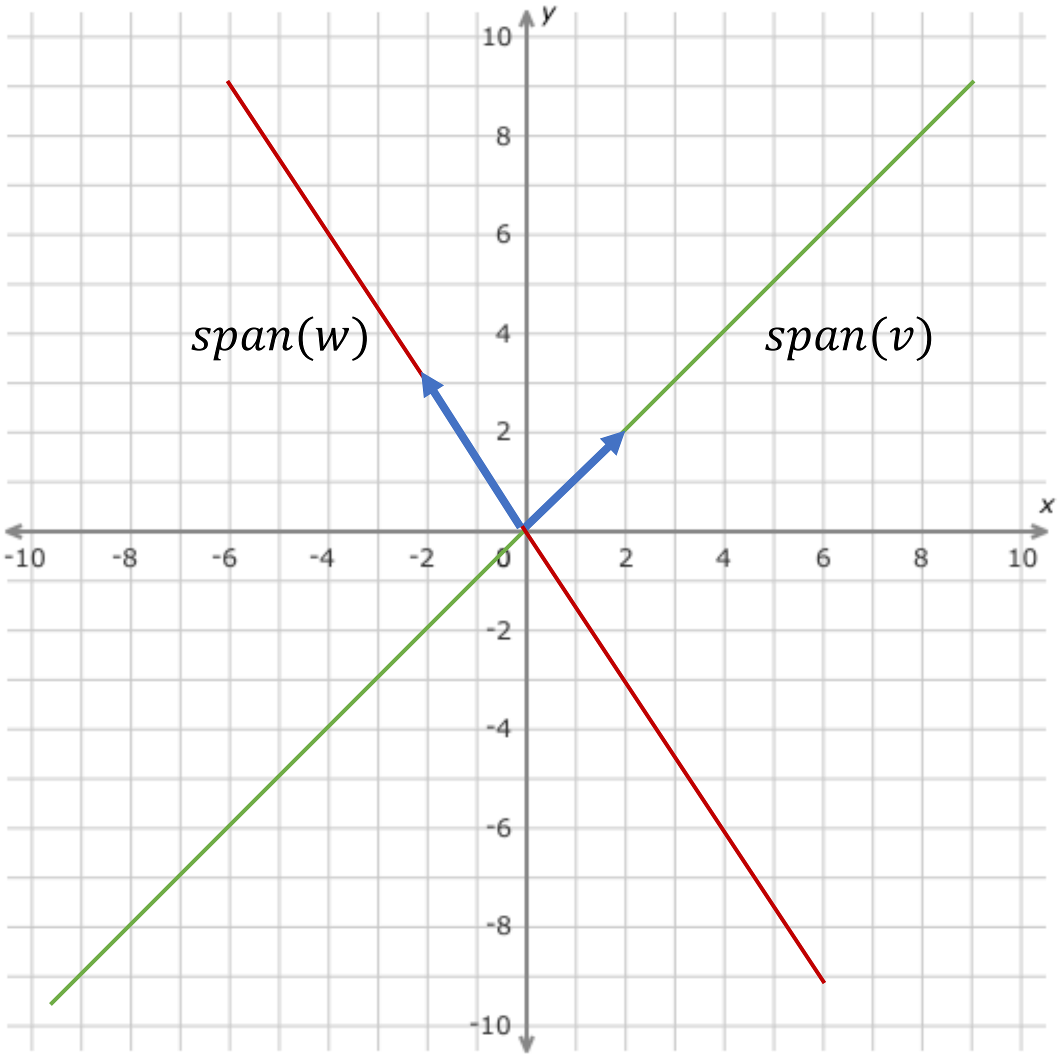
Visualizations in $\mathbb{R}^2$



Visualizations in $\mathbb{R}^2$

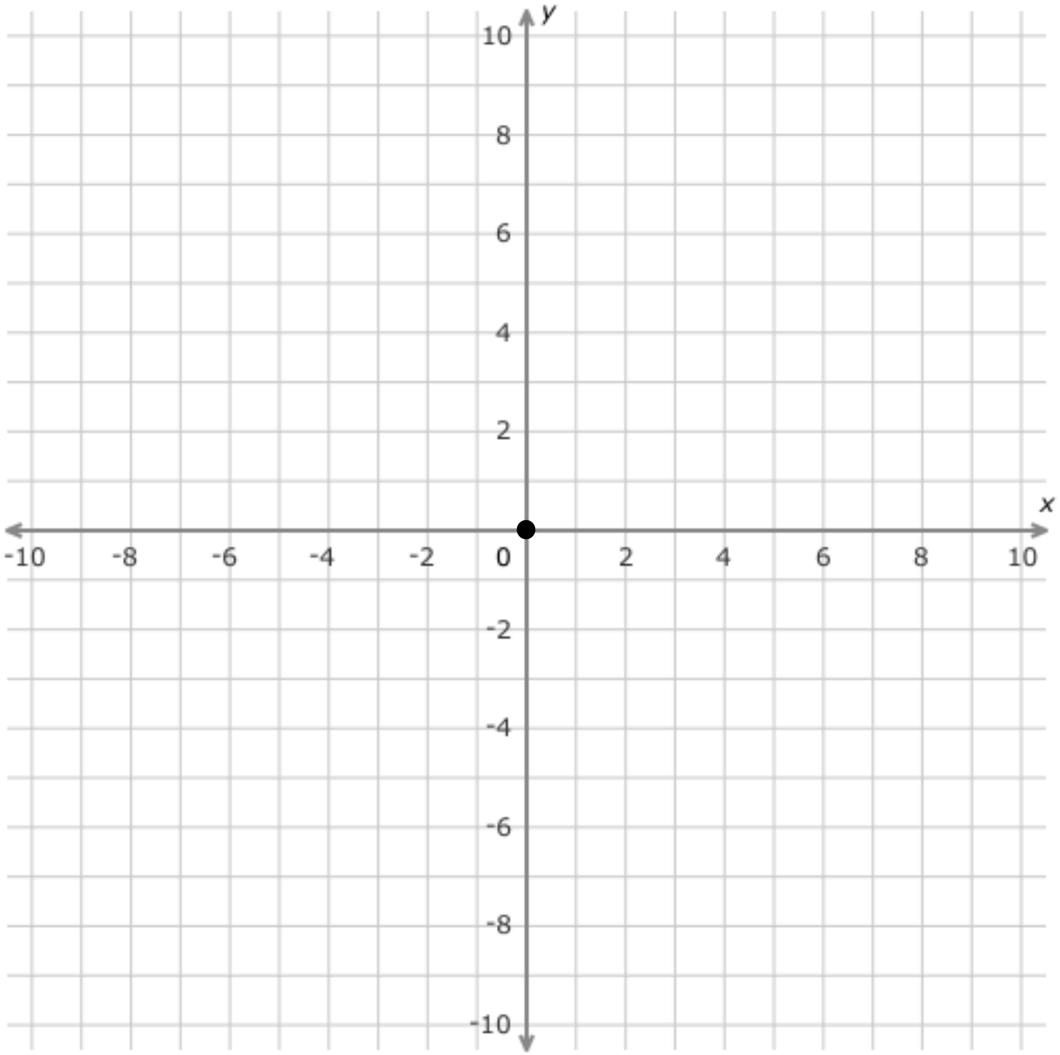


Visualizations in $\mathbb{R}^2$



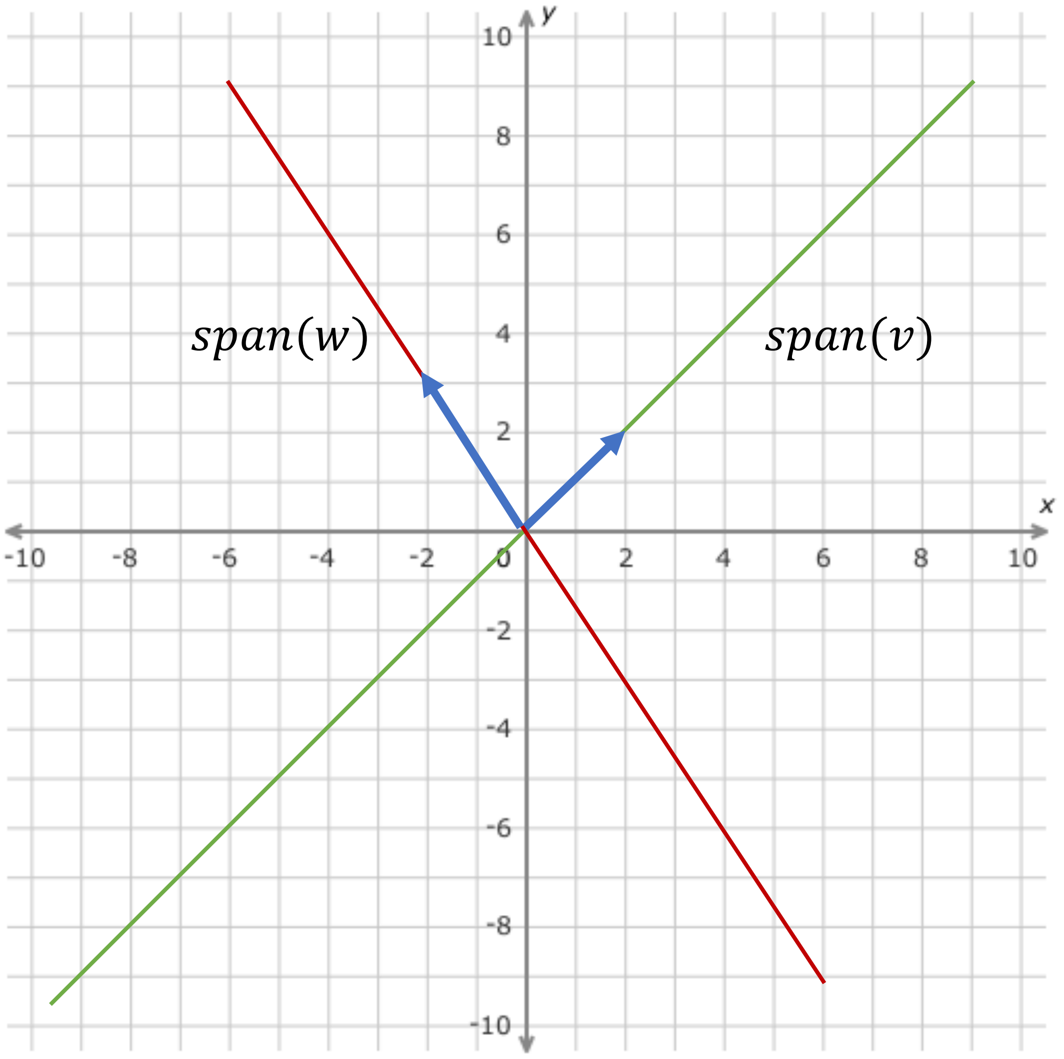
$span(v) \cup span(w)$

Visualizations in $\mathbb{R}^2$

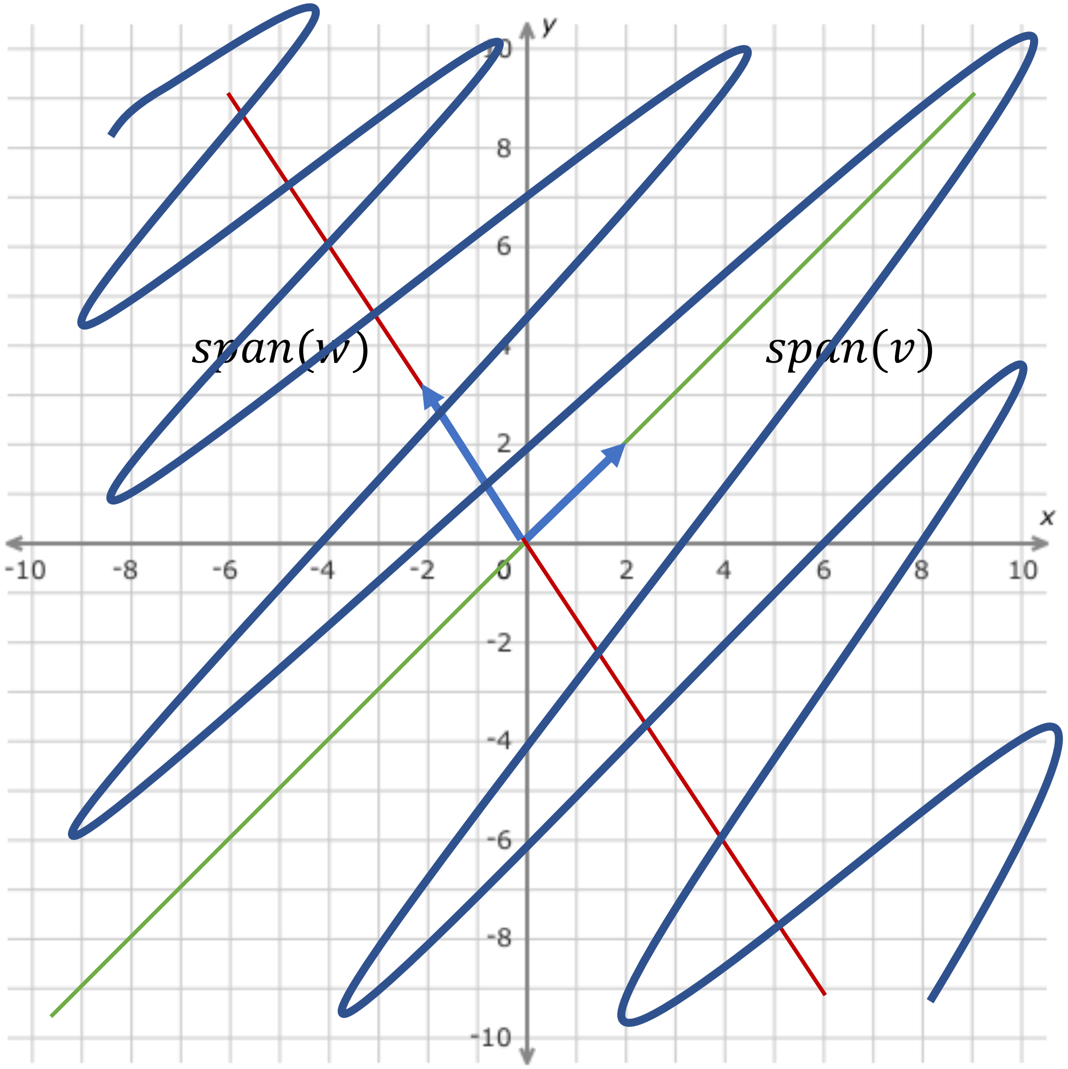


$span(v) \cap span(w)$

Visualizations in $\mathbb{R}^2$

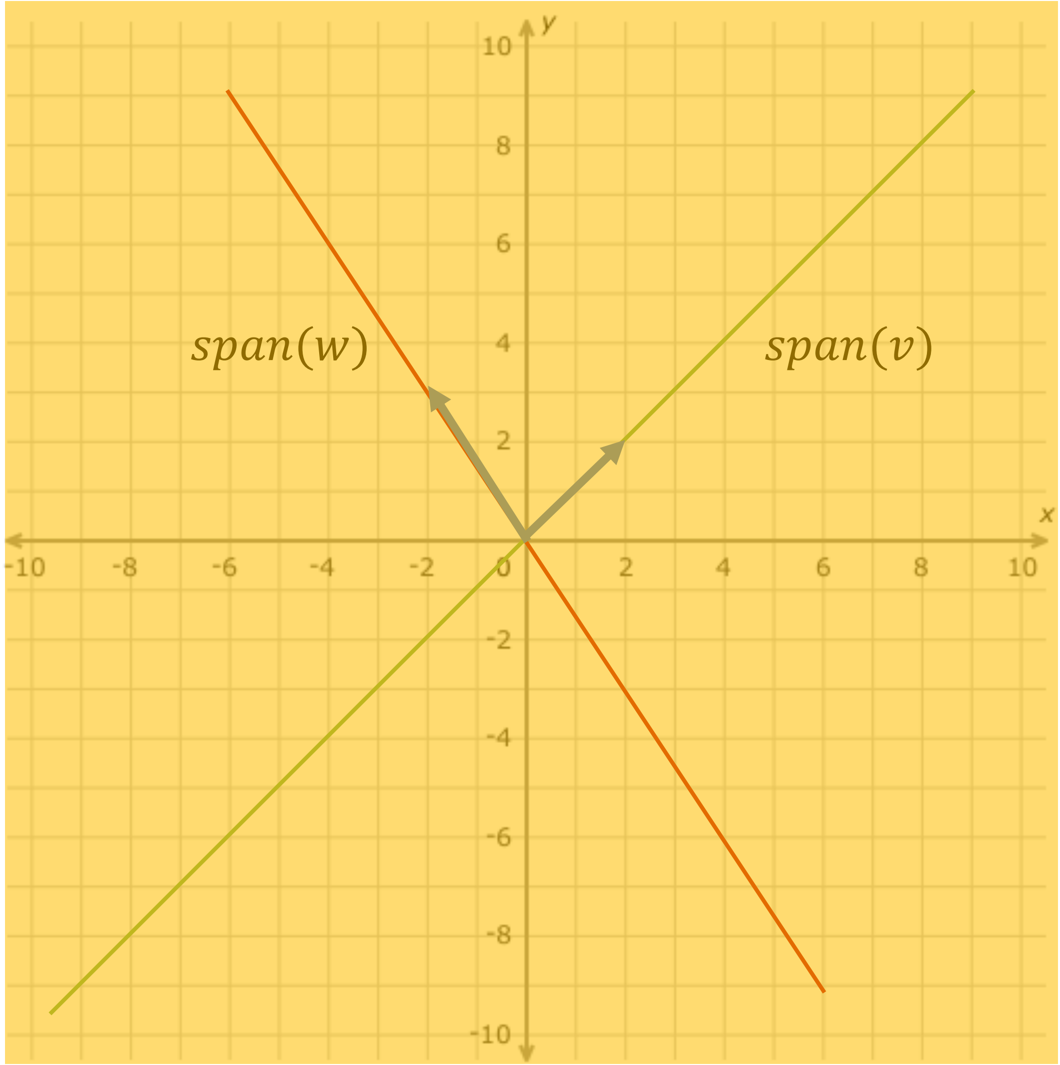


Visualizations in $\mathbb{R}^2$



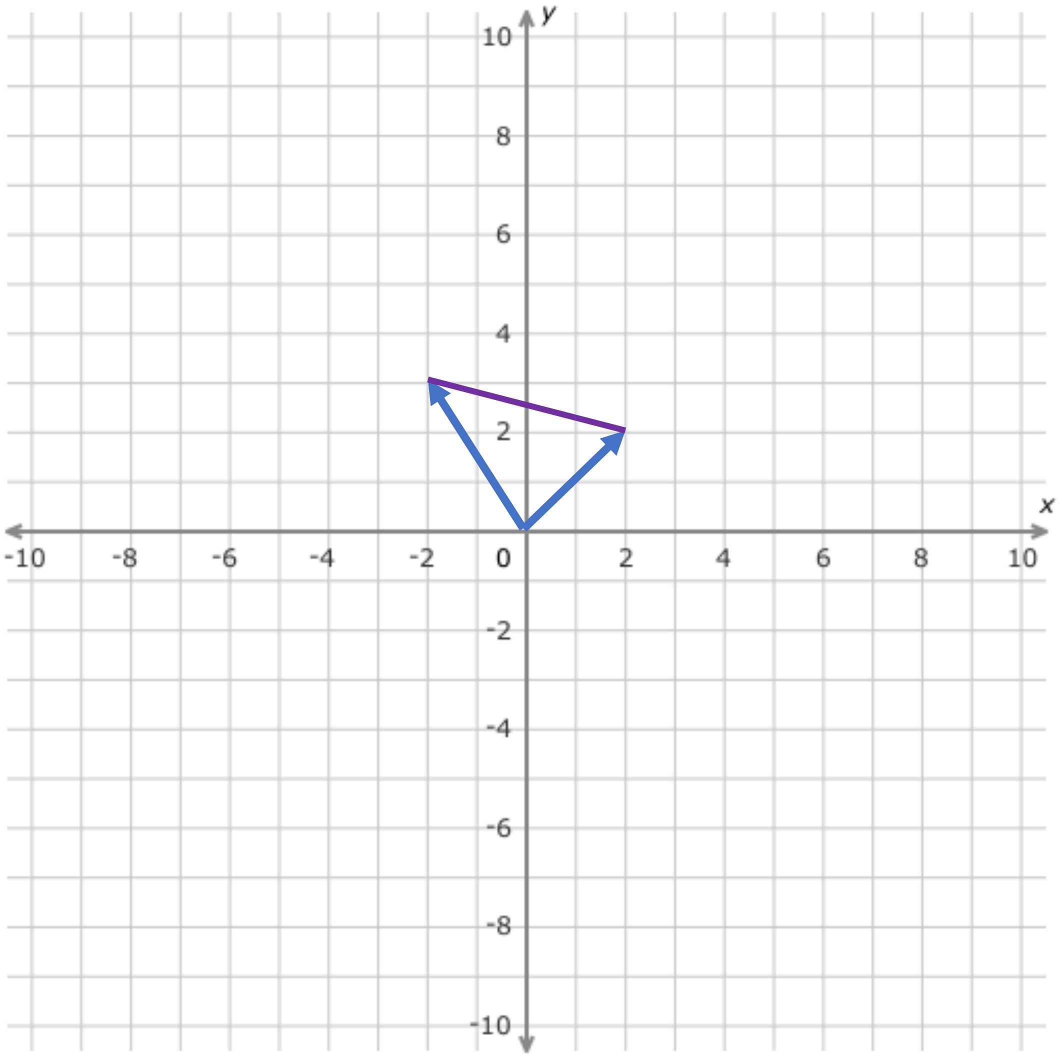
$span(v, w)$

Visualizations in $\mathbb{R}^2$



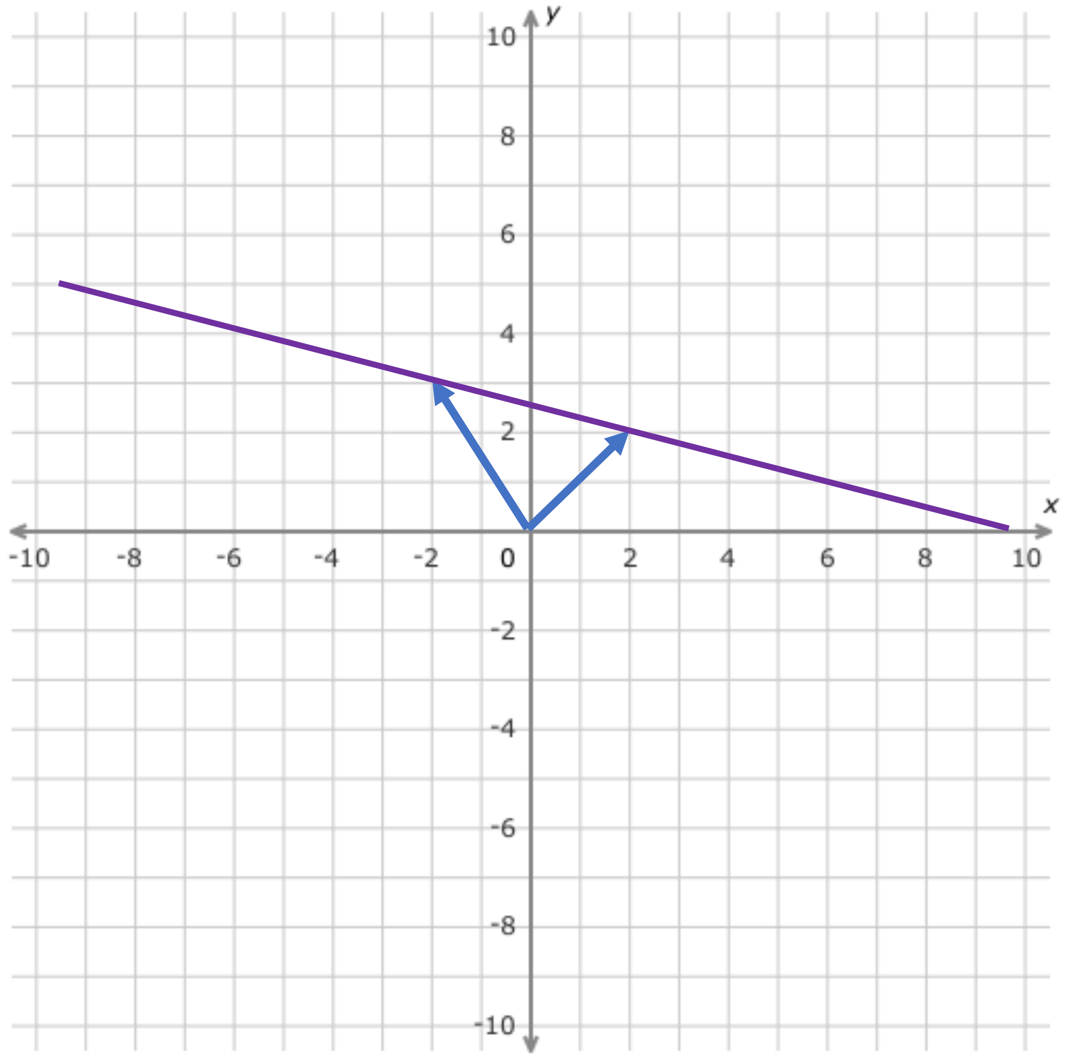
$span(v, w)$

Visualizations in $\mathbb{R}^2$



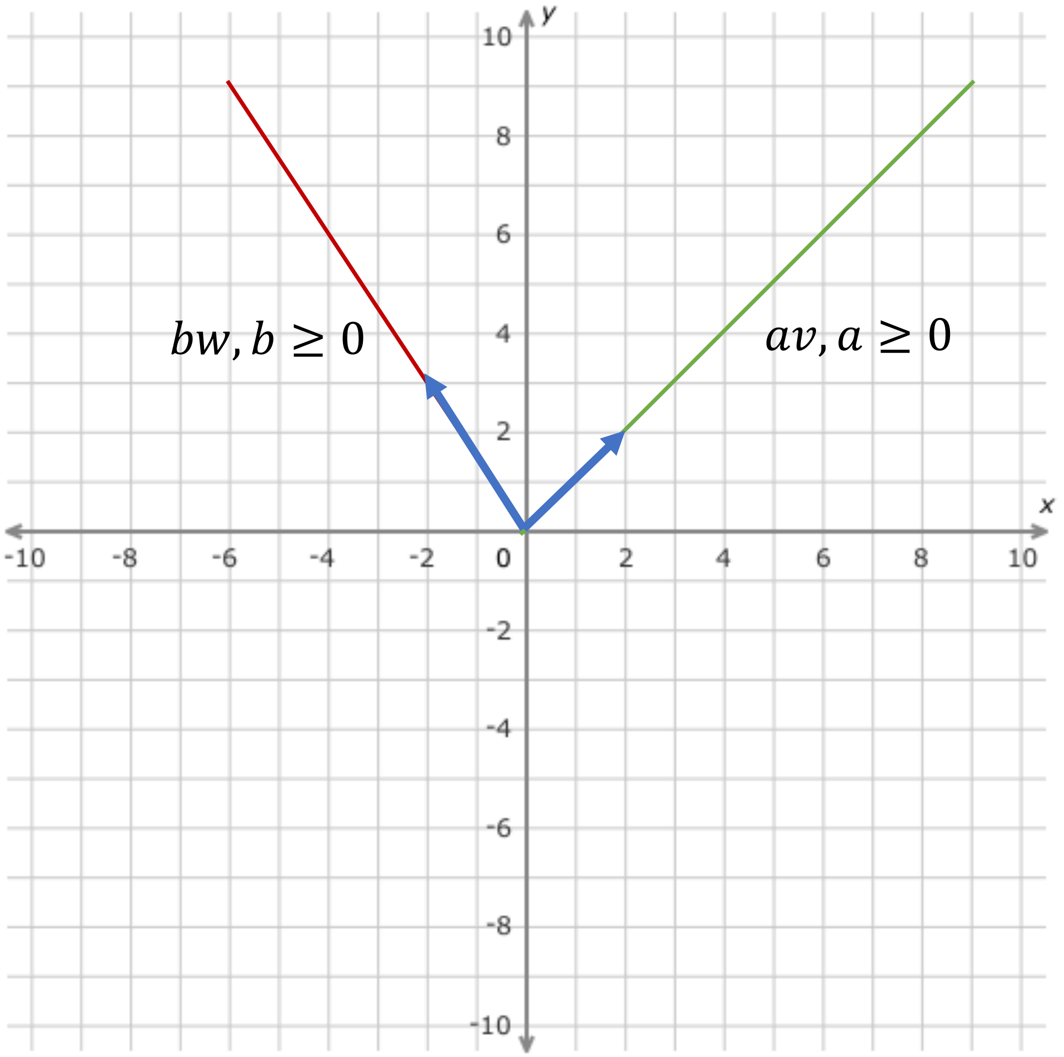
$$tv + (1 - t)w, t \in [0,1]$$

Visualizations in $\mathbb{R}^2$

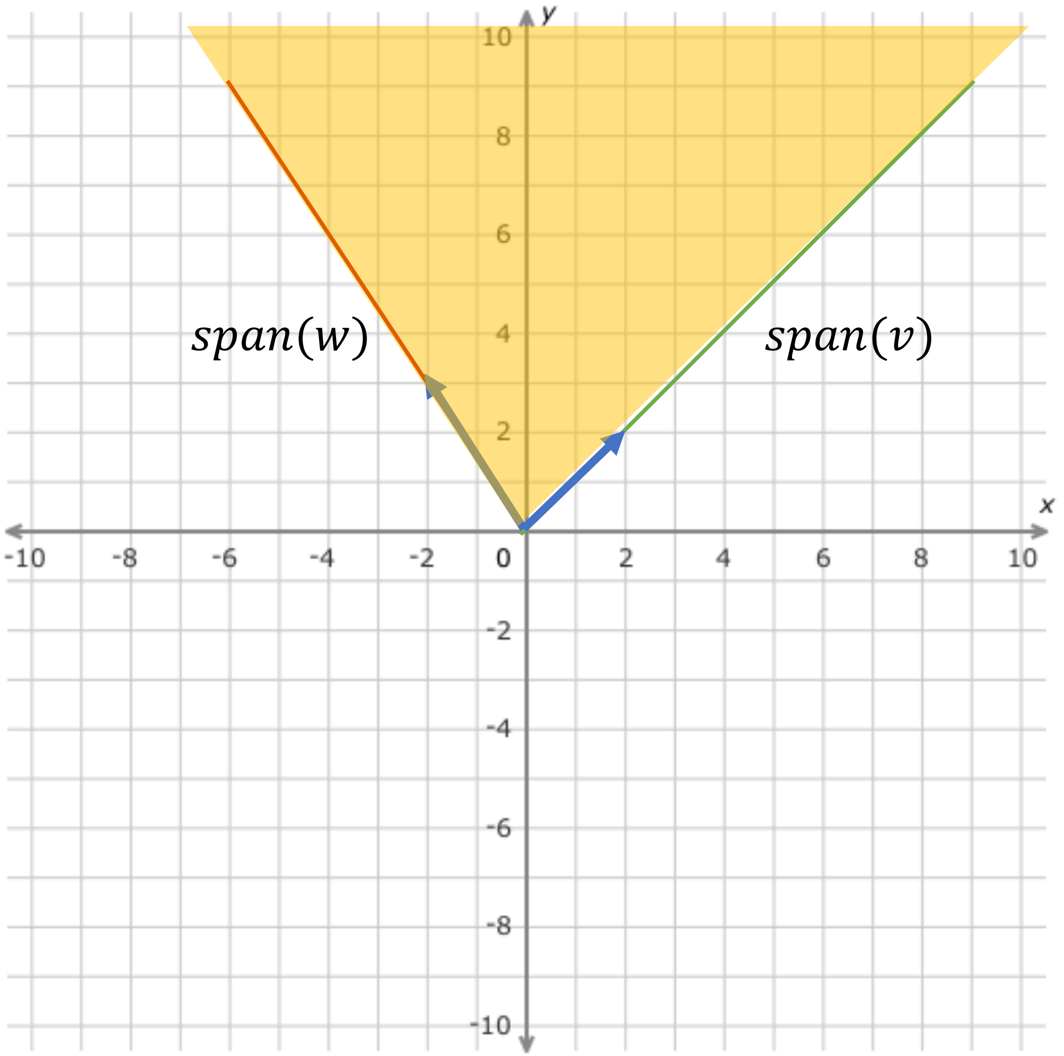


$$tv + (1 - t)w, t \in \mathbb{R}$$

Visualizations in $\mathbb{R}^2$

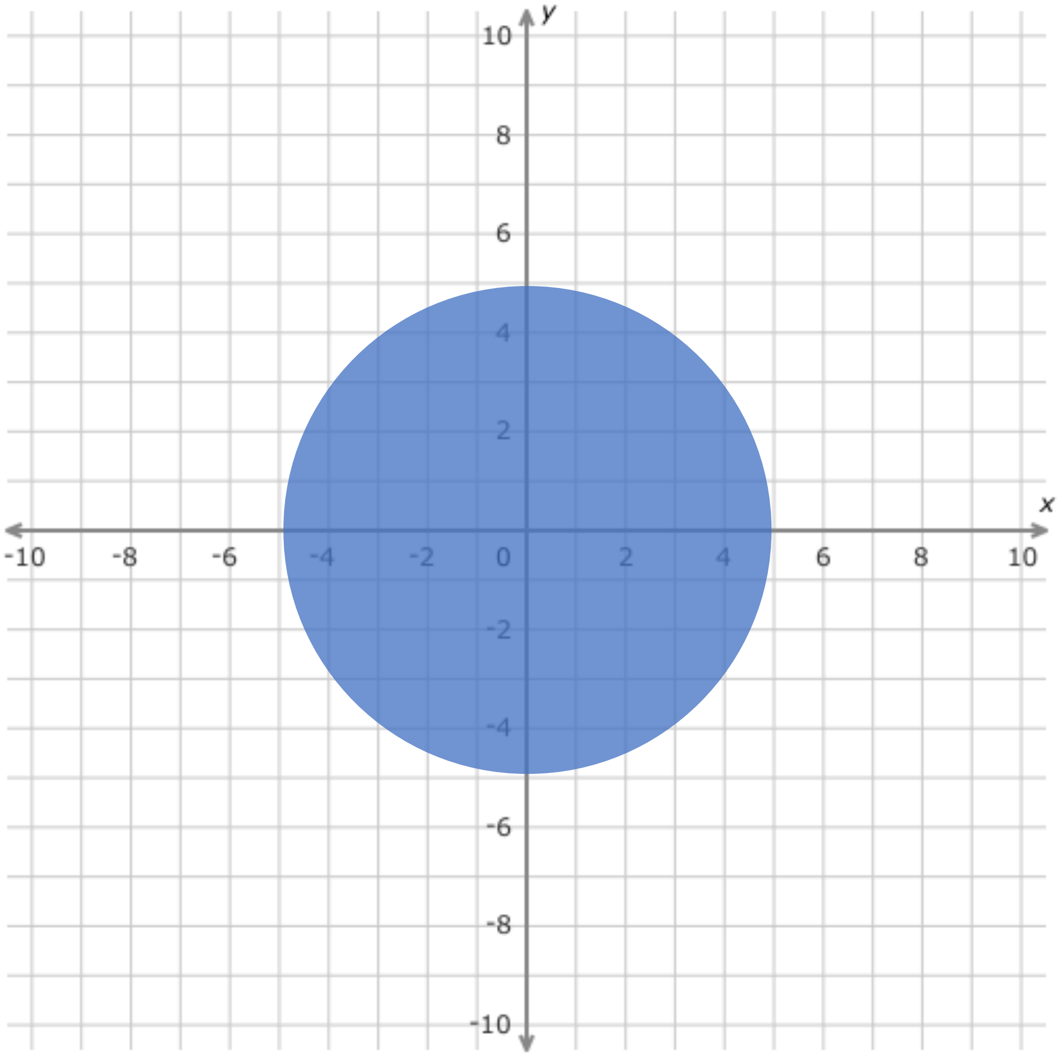


Visualizations in $\mathbb{R}^2$



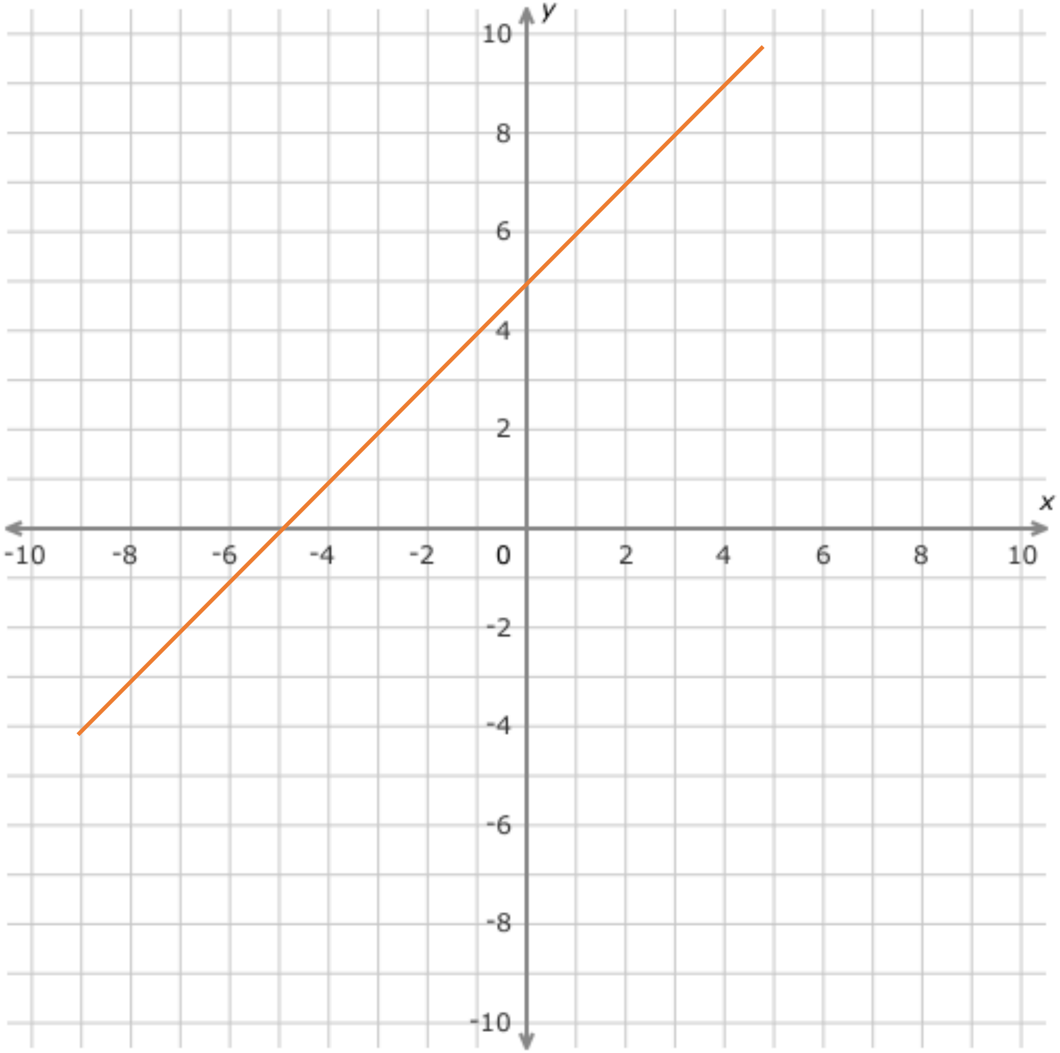
$$av + bw \mid a.b \geq 0$$

Visualizations in $\mathbb{R}^2$



$$a^2 + b^2 \leq 25$$

Visualizations in $\mathbb{R}^2$



$(a, a+5) \quad a \in \mathbb{R}$

Linear Independence, Span, Basis and Dimension

1. Let $V \coloneqq \mathbb{R}^{n\times n}$ be the space of $n\times n$ matrices. Prove that V is a real vector space. Find the dimension of V. Let U be the the space of $n\times n$ diagonal matrices. Is U a subspace of V? What is the dimension of U?
2. Let $v_1, v_2, v_3, v_4(\text{all distinct}) \in \mathbb{R}^3$ and $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for dim(Span(v_1, v_2, v_3, v_4))? No proof necessary
3. True or False: If B is a basis of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$, then a subset of B is the basis of W
4. Consider the non-empty set of functions $V \coloneqq \{p: \mathbb{R} \to \mathbb{R} \mid p(x) = \Sigma_{k=0}^{n} a_k x^k \ for\ a_k \in \mathbb{R}, and\ x \in \mathbb{R} \text{ is a constant}\}$. Define an addition operation $+: V\times V \to V$ and a scalar multiplication operation $\cdot : \mathbb{R}\times V \to V$ such that the triple $(V, +, \cdot)$ is a real vector space. Find a basis of this vector space and deduce its dimension
5. Suppose $(v_1, v_2, \dots, v_m) \in \mathbb{R}^n$ be linearly dependent. Prove that for x $\in span(v_1, v_2, \dots, v_m)$, there exist infinitely many $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m$ such that $x = \Sigma \alpha_i v_i$

1. V satisfies the 3 conditions for a set to be a vector space:
   - If matrix $A \in V$, and matrix $B \in V$, then $A + B \ \$\backslash\mathrm{in}\ \mathbb{R}^{n\times n} \Rightarrow A \in V$
   - If matrix $A \in V$, and $\alpha \in \mathbb{R}$, then $\alpha A \in \mathbb{R}^{n\times n} \Rightarrow \alpha A \in V$
   - Zero matrix of size $n \times n \in V$

   The dimension of a vector space is defined as the size of it's basis. The basis of the space of matrices of size $n \times n$ is:

$$\begin{bmatrix} 1 \ldots 0 \\ 0 \ldots 0 \\ \vdots \\ 0 \ldots\ 0 \end{bmatrix}, \begin{bmatrix} 0\ 1\ 0 \ldots 0 \\ 0\ 0\ 0 \ldots 0 \\ \vdots \\ 0\ 0\ 0 \ldots\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \ldots 0 \\ \vdots \\ 0\ \ldots 0 \\ 0 \ldots 1 \end{bmatrix}$$

   So like we have a canonical basis for vectors, this is a canonical basis for matrices (Show that this in fact is a basis of V by proving that it spans V and is linearly independent). Since we have n^2 elements in the basis, the dimension of V is n^2

Solutions

1. U satisfies the 3 conditions for a subset to be a subspace:
   - If matrix $A \in U$, and matrix $B \in U$, then $A + B$ (still diagonal) $\in U$
   - If matrix $A \in V$, and $\alpha \in \mathbb{R}$, then αA(still diagonal) $\in U$
   - Zero matrix of size $n \times n \in U$

   Similar to V, the basis of the space of diagonal matrices of size $n \times n$ is:

$$\begin{bmatrix} 1 \ldots 0 \\ 0 \ldots 0 \\ \vdots \\ 0 \ldots \ 0 \end{bmatrix}, \begin{bmatrix} 0\ 0\ 0 \ldots 0 \\ 0\ 1\ 0 \ldots 0 \\ \vdots \\ 0\ 0\ 0 \ldots \ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \ldots 0 \\ \vdots \\ 0 \ \ldots 0 \\ 0 \ldots 1 \end{bmatrix}$$

   Since we have n elements in the basis, the dimension of U is n

Solutions

2. Any set of vectors (≥ 3 in size) $\in \mathbb{R}^3$ can't have dimension > 3. So, the maximum dimension Span(v_1, v_2, v_3, v_4)) can have is 3. But if $C_1 \subset Span(C_2)$ or $C_2 \subset Span(C_1)$, then the dimension is 2

3. False. Consider n=2, B={(1,0),(0,1)} and W=Span((1,1))

4. V is a space of polynomials of degree at most n. Any polynomial in this space can be constructed by using the following set of vectors:

$$1, x, x^2, x^3, \dots x^n$$

Example (for $n > 3$):

$$2x^2 + 4 = 0 \cdot x^n + 0 \cdot x^{n-1} + \dots + 2 \cdot x^2 + 0 \cdot x + 4 \cdot 1$$

And this set of vectors is also linearly independent. So, this is the basis of V and the dimension is n+1

Solutions

5. $(v_1, v_2, \dots, v_m) \in \mathbb{R}^n$ are linearly dependent

$$\Rightarrow \beta_1 v_1 + \cdots + \beta_m v_m = 0 \; for \; some \; (\beta_1, \dots, \beta_m) \neq \mathbf{0}$$

$x \in span(v_1, \dots, v_m) \Rightarrow \gamma_1 v_1 + \cdots + \gamma_m v_m = x \; for \; \gamma_i \in \mathbb{R} \; \forall \; i$. Then we have

$\Rightarrow x = \gamma_1 v_1 + \cdots + \gamma_m v_m + r(\beta_1 v_1 + \cdots + \beta_m v_m) \; for \; some \; r \in \mathbb{R}$

$\Rightarrow \alpha_i = \gamma_i + r \cdot \beta_i$

Thus, depending on r, we may have infinitely many $\alpha' s$