Recitation Week 11 Solutions

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Convexity

1. For $f: \mathbb{R}^n \to \mathbb{R}$, define epigraph $epi(f) \subset \mathbb{R}^{n+1}$ to be set of all the points above the graph of f:

$$epi(f) = \{(x, t) \in \mathbb{R}^{n+1}: t \geq f(x)\}$$

   Prove that f is convex if and only if $epi(f)$ is convex

2. Let $f: \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = x^T Ax$ for some symmetric matrix $A \in \mathbb{R}^{n \times n}$. Give conditions on A so that 0 is the global minimizer of f

Convexity

1. a. If f is convex, then $epi(f)$ is convex

Proof: If epi(f) is convex, then for any two points (x_1, t_1) and $(x_2, t_2) \in epi(f)$, $\theta(x_1, t_1) + (1-\theta)(x_2, t_2) \in epi(f)$

$$\Rightarrow (\theta x_1 + (1-\theta)x_2, \theta t_1 + (1-\theta)t_2) \in epi(f)$$

$$\Rightarrow \theta t_1 + (1-\theta)t_2 \geq f(\theta x_1 + (1-\theta)x_2) \ (To\ prove)$$

Now it's given that f is convex,

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)$$

$$\Rightarrow f(\theta x_1 + (1-\theta)x_2) \leq \theta t_1 + (1-\theta)t_2$$

which is exactly what we wanted to prove

Convexity

1. b. If $\mathrm{epi}(f)$ is convex, then f is convex

   Proof: If epi(f) is convex, then for any two points $(x_1, f(x_1))$ and $(x_2, f(x_2)) \in epi(f)$,

$$\theta(x_1, f(x_1)\,) + (1-\theta)(x_2, f(x_2)) \in epi(f)$$

$$\Rightarrow \big(\theta x_1 + (1-\theta)x_2, \theta f(x_1) + (1-\theta) f(x_2)\big) \in epi(f)$$

$$\Rightarrow \theta f(x_1) + (1-\theta) f(x_2) \geq \theta x_1 + (1-\theta) x_2$$

   This proves that x is convex

Convexity

2. If 0 is the global minimizer of f, then $f(x) \geq f(0) \; \forall \; x$

$$\Rightarrow f(x) \geq 0 \; \forall \; x$$

$$\Rightarrow x^T A x \geq 0 \; \forall \; x$$

Thus, the only condition on A is that it should be positive semidefinite

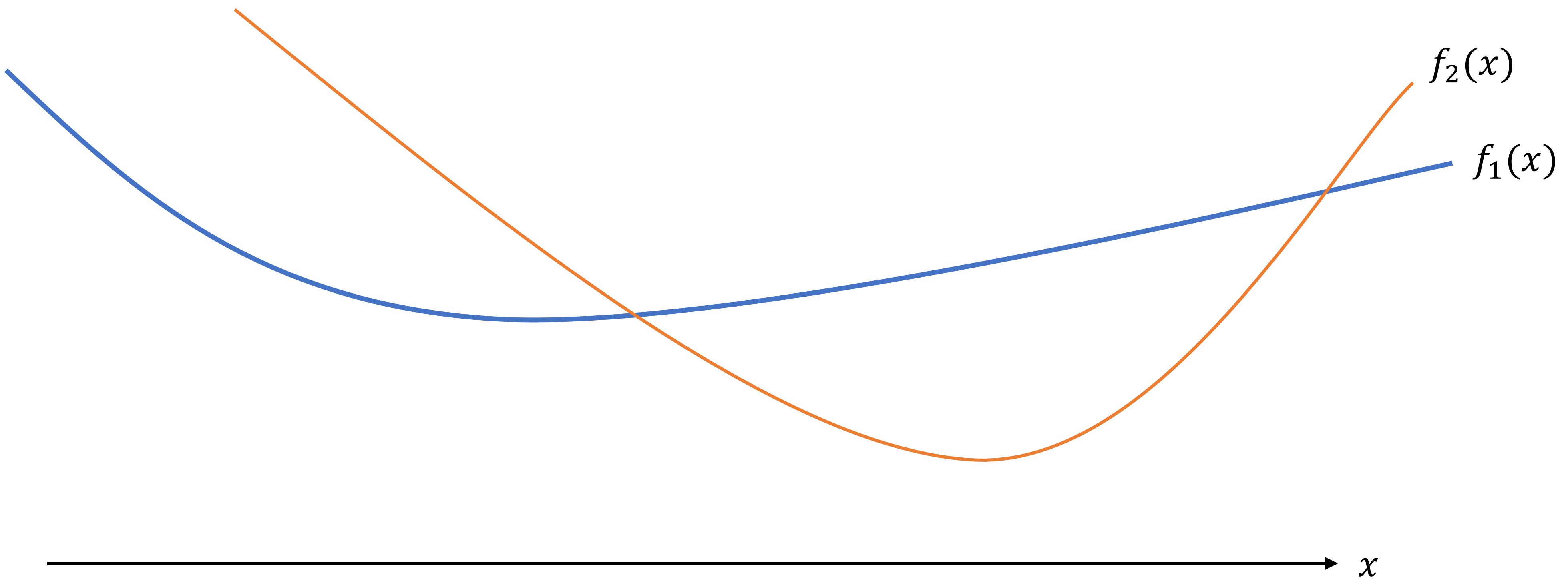
Convexity

1. True or False:
    1. If f has only 1 global minima and no local minima, then f is convex
    2. Linear combination of two convex functions is convex
    3. Convex functions are differentiable at all points
    4. Norms are convex functions
    5. If f is convex, then $g(x) = f(Ax - b)$ is also convex in x
    6. Sum of a non-convex function (like $\cos(x)$) with another function can never be convex
    7. Union of convex sets is convex
    8. Intersection of convex sets is convex
    9. Maximum of two convex functions is convex
    10. If f is convex, f^n is also convex for $n \in \mathbb{N}$

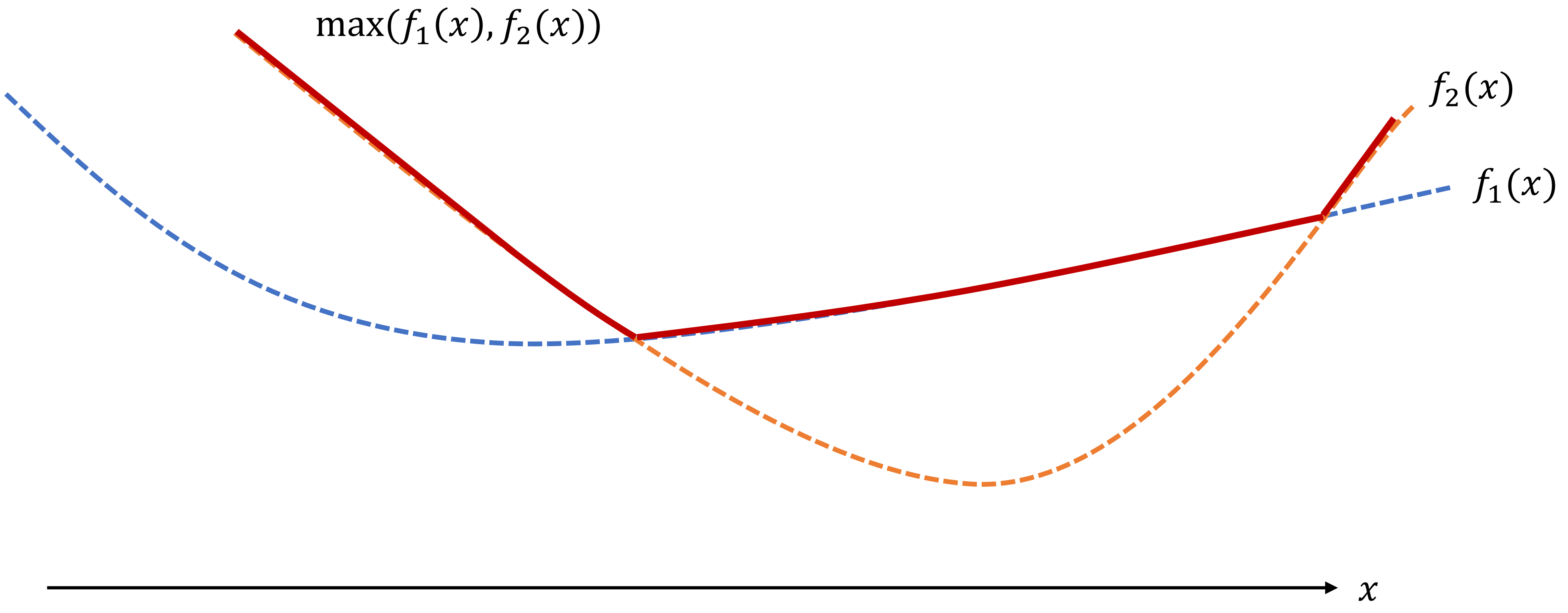
Convexity

1. True or False:
    1. False
    2. False
    3. False
    4. True
    5. True
    6. False
    7. False
    8. True
    9. True
    10. False

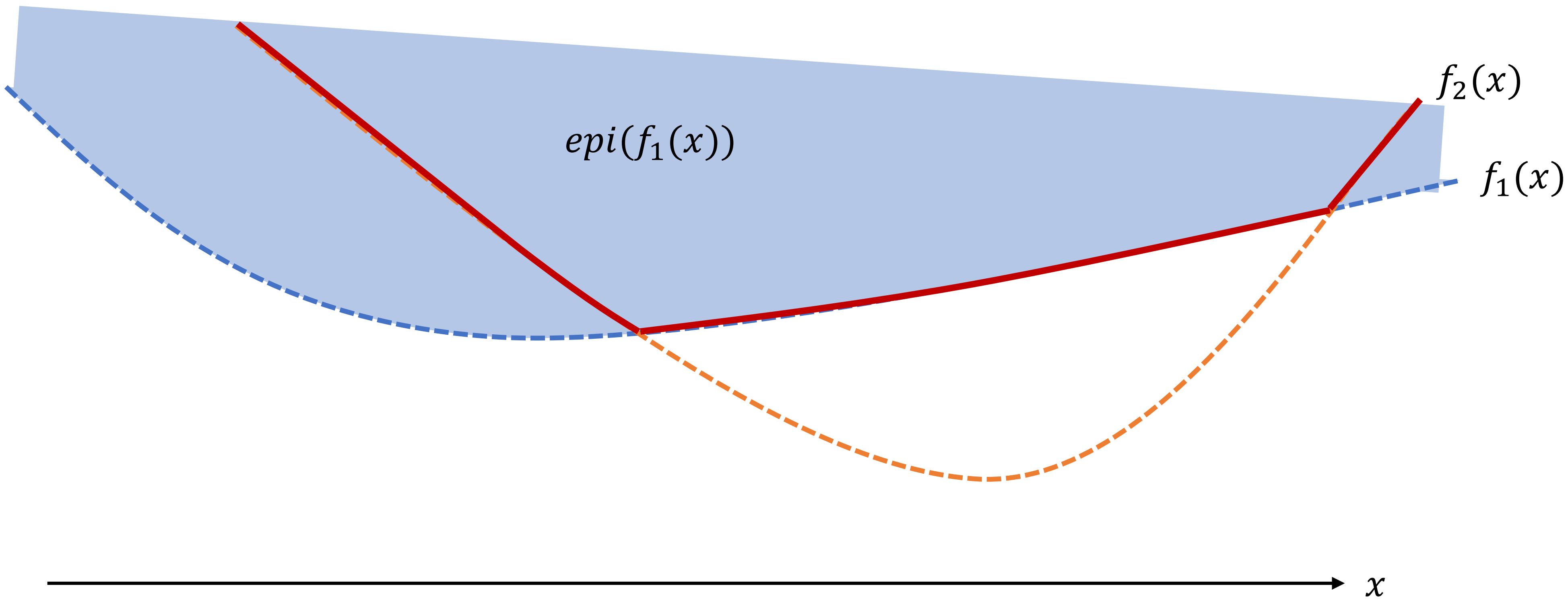
Convexity



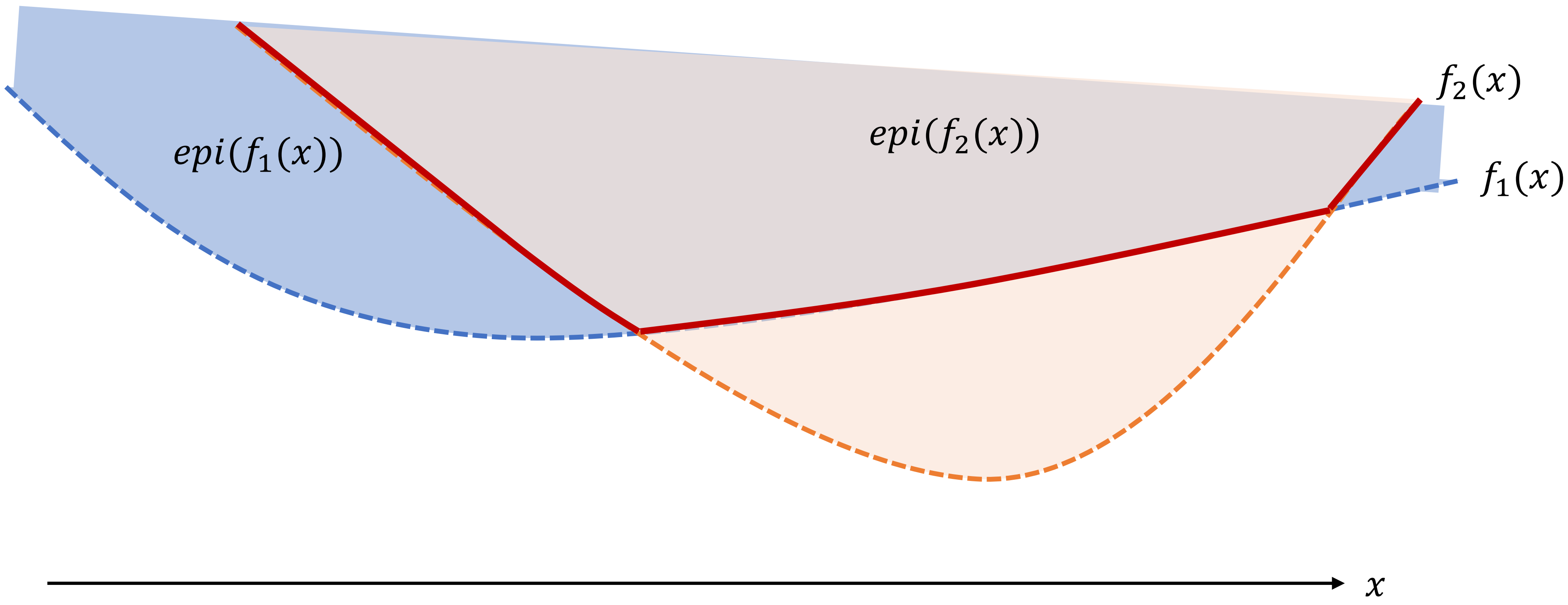
Convexity



Convexity



Convexity



Multivariable differentiability

- We say a function $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - g^T h}{||h||_2} = 0$$

for some $g \in \mathbb{R}^n$. Here, h takes values in $\mathbb{R}^n$. If it exists, this g is unique and is called the gradient of f at x with notation $g = \nabla f(x)$

- The directional derivative $f'(x; v)$ of $f: \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in the direction $v \neq 0$ is defined by

$$f'(x; v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

assuming the limit exists. If f is differentiable, $f'(x; v) = \nabla f(x)^T v$ (Prove it!)

Multivariable differentiability

1. Let $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\nabla f(x) \neq 0$. Compute the direction of steepest descent and steepest ascent:

$$arg \min_{||v||=1} f'(x; v) \quad \& \quad arg \max_{||v||=1} f'(x; v)$$

Solution:

$$f'(x; v) = \nabla f(x)^T v =< f(x), v >$$

From HW we know that for fixed u and $||v|| = 1, < u, v >$ is maximized by $v = \frac{u}{||u||}$ and minimized by $v = \frac{-u}{||u||}$

Thus, $arg \min_{||v||=1} f'(x; v) = -\frac{\nabla f(x)}{||\nabla f(x)||}$ and $arg \max_{||v||=1} f'(x; v) = \frac{\nabla f(x)}{||\nabla f(x)||}$