Recitation Week 10 Solutions

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Rank k approximation

1. Let $A \in \mathbb{R}^{m \times n}$ be a matrix such that using SVD, $A = U\Sigma V^T$. Let U_r and V_r denote the first r columns of U and V respectively
   i. Let $A \in \mathbb{R}^{m \times n}$ be a rank one matrix. Find the value of the square of the its first singular value in terms of entries of A
   ii. Let's say we have a function f to measure distance between two matrices such that

$$f: X, Y \rightarrow \mathbb{R}\,; \quad \mathrm{X}, \mathrm{Y} \in \mathbb{R}^{m \times n}$$

   How would you formulate the optimization problem to find a rank k matrix A_k which is at minimal distance to A?

Rank k approximation

1. Any rank one matrix can be expressed as xy^T where x is basis of Image(A) and y is the basis of $Im(A^T)$. Now, using SVD, we can write A as $A = \sigma_1 u_1 v_1^T$. We know that left singular vectors belong to Im(A) and right singular vectors belong to $Im(A^T)$. This means u_1 is an orthonormal vector in Image(A) and v_1 is an orthonormal vector in $Im(A^T)$

$$u_1 = \frac{x}{||x||} \; amd \; v_1 = \frac{y}{||y||}$$

Thus, $\sigma_1 = ||x||||y||$

ii. Optimization problem: $minimize \; f(A, A_k)$ with the constraint that $\text{rank}(A_k) = k$. The solution to this comes out to be $A_k = U_k \Sigma_k V_k^T$ where U_k and $\mathrm{V_k}$ consists of first k columns of U and V respectively, and Σ_k is a square matrix with largest k singular value of A in descending order

SVD

1. Let $A \in \mathbb{R}^{m \times n}$ be a rectangular matrix such that using SVD, $A = U\Sigma V^T$ and let $rank(A) = r$. Let U_r and V_r denote the first r columns of U and V respectively
   1. Show that A can also be expressed as $A = XY^T$ where $X \in \mathbb{R}^{m \times r}$ and Y $\in \mathbb{R}^{n \times r}$ such that X and Y have orthogonal columns respectively. Is this factorization unique?
   2. Computationally, what's the advantage of expressing $A \in \mathbb{R}^{m \times n}$ as $\mathrm{A} = XY^T$

SVD

1. Let $A \in \mathbb{R}^{m \times n}$ be a rectangular matrix such that using SVD, $A = U\Sigma V^T$ and let $rank(A) = r$. Let U_r and V_r denote the first r columns of U and V respectively

   1. $A = U_{\mathrm{r}}\Sigma_{\mathrm{r}}V_r^T$

   $$\Rightarrow A = (U_r\Sigma_r)V_r^T = XY^T$$

   Where $X = U_r\Sigma_r$ and $Y = V_r$. This factorization is not unique because we can also factorize it as $X = U_r$ and $Y = V_r\Sigma_r$

   2. The advantage of matrix factorization is that it saves memory i.e. we need less numbers ((m+n)*r compared to m*n) to store a matrix now