Concept check Solutions - Week 2

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Concept Check

1. Suppose U is a subspace of V with $U \neq V$. Let $S: U \rightarrow W$ be a linear transformation and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T: V \rightarrow W$ such that $\mathrm{T(v)} = \begin{cases} Sv \ if \ v \in U \\ 0 \ if \ v \in \mathrm{V} \ and \ v \notin U \end{cases}$. Prove that T is not a linear map on V
2. If $v_1, \ldots, v_m$ is a list of linearly independent vectors in $\mathbb{R}^n$, then for any $a \neq 0$ and index i, must $v_1, \ldots, v_{i-1}, av_i, \ldots, v_m$ be also linearly independent? Must they have the same span?
3. If $v_1, \ldots, v_m$ is a list of linearly independent vectors in $\mathbb{R}^n$, then for any $\mathrm{b} \neq 0$ and index $i \neq j$, must $v_1, \ldots, v_{i-1}, v_i + bv_j, \ldots, v_m$ be also linearly independent? Must they have the same span?
4. Find two linearly independent vectors in $\mathbb{R}^4$ on the plane $x + 2y - 3z - t = 0$. Then find three independent vectors? What about four?

($*$) marked problems are not as easy as others. Treat them as the most optional (optional-est?) thing among the whole course

Concept Check

1. Suppose U is a subspace of V with $U \neq V$. Let $S: U \rightarrow W$ be a linear transformation and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T: V \rightarrow W$ such that $\mathrm{T}(\mathrm{v}) = \begin{cases} Sv \; if \; v \in U \\ 0 \; if \; v \in \mathrm{V} \; and \; v \notin U \end{cases}$. Prove that T is not a linear map on V

Solution:

$$Let \; v \in V \; such \; that \; v \notin U \; and \; u \in U$$
$$\Rightarrow v + u \in V \; and \; v + u \notin U$$
$$\Rightarrow T(v + u) = 0$$

But

$$T(v) = 0 \; and \; T(u) = Su$$
$$\Rightarrow T(v) + T(u) = Su$$
$$\Rightarrow T(u + v) \neq T(u) + T(v)$$

This proves that T is not a linear transformation

Concept Check

2. If $v_1, \ldots, v_m$ is a list of linearly independent vectors in $\mathbb{R}^n$, then for any $a \neq 0$ and index i, must $v_1, \ldots, v_{i-1}, av_i, \ldots, v_m$ be also linearly independent? Must they have the same span?

Solution: To check if the given set is linearly independent or not:

$$\alpha_1 v_1 + \cdots + \alpha_i a v_i + \cdots + \alpha_m v_m = 0$$

$$\Rightarrow \alpha_1 v_1 + \cdots + (\alpha_i a) v_i + \cdots + \alpha_m v_m = 0$$

All coefficients must be zero given that $v_1, . . , v_m$ are linearly independent

$$\Rightarrow \alpha_1 = \cdots = a\alpha_i = \cdots = \alpha_m = 0$$

But we know that $a \neq 0$

$$\Rightarrow \alpha_i = 0 \ \forall \ i$$

Thus, the set $v_1, \ldots, v_{i-1}, av_i, \ldots, v_m$ is linearly independent

By similar method you can prove that they have the same span

Concept Check

3. If $v_1, \dots, v_m$ is a list of linearly independent vectors in $\mathbb{R}^n$, then for any b $\neq 0$ and index $i \neq j$, must $v_1, \dots, v_{i-1}, v_i + bv_j, \dots, v_m$ be also linearly independent? Must they have the same span?

Solution: To check if the given set is linearly independent or not:

$$\alpha_1 v_1 + \cdots + \alpha_i(v_i + bv_j) + \cdots + \alpha_m v_m = 0$$

$$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_i v_i + \cdots + v_j(\alpha_j + b\alpha_i) + \cdots + \alpha_m v_m = 0$$

All coefficients must be zero given that $v_1, . . , v_m$ are linearly independent

$$\Rightarrow \alpha_1 = \cdots = \alpha_i = \cdots = \alpha_j + b\alpha_i = \cdots = \alpha_m = 0$$

But we know that b$\neq 0$

$$\Rightarrow \alpha_i = 0 \,\forall\, i$$

Thus, the set $v_1, \dots, v_{i-1}, v_i + bv_j, \dots, v_m$ is linearly independent

By similar method you can prove that they have the same span

Concept Check

4. Find two linearly independent vectors in $\mathbb{R}^4$ on the plane $x + 2y - 3z - t = 0$. Then find three independent vectors? What about four?

Solution:

For problems like this one, I start by trying to find the general form of vectors in the space. The vectors lying on this plane will have the form $\begin{bmatrix} x \\ y \\ z \\ x + 2y - 3z \end{bmatrix}$

2 independent vectors would be: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

2 independent vectors would be: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$. The last coordinate of the vector is dependent on the first 3 coordinates (which are independent of each other). So I cannot find 4 independent vectors on the plane

Concept Check

5. Suppose $v_1, \dots, v_6$ are six vectors in $\mathbb{R}^4$. These vectors:
   - (do)(do not)(might not) span $\mathbb{R}^4$
   - (are)(are not)(might be) linearly independent
   - Any 4 of those vectors (are)(are not)(might be) a basis for $\mathbb{R}^4$
6. Suppose S is a 5 dimensional subspace of $\mathbb{R}^6$. True or false (example if false):
   - Every basis for S can be extended to a basis for $\mathbb{R}^6$ by adding one more vector
   - Every basis for $\mathbb{R}^6$ can be reduced to a basis for S by removing one vector
7. ($*$)We can think of a permutation of n elements as a linear transformation $P: \mathbb{R}^n \to \mathbb{R}^n$ that permutes the basis elements $e_1, \dots, e_n$. In that case it is an $n \times n$ matrix with only 0's and 1's as entries and such that every row and every column have exactly one entry being 1. Is it true that for any n and any permutation P there exists an integer $k \geq 1$ such that $P^k = I$, where I is the identity matrix. Justify your answer.

($*$) marked problems are not as easy as others. Treat them as the most optional (optional-est?) thing among the whole course

Concept Check

5. Suppose $v_1, \ldots, v_6$ are six vectors in $\mathbb{R}^4$. These vectors:

    i. (do)(do not)(might not) span $\mathbb{R}^4$
    ii. (are)(are not)(might be) linearly independent
    iii. Any 4 of those vectors (are)(are not)(might be) a basis for $\mathbb{R}^4$

Solution:

i. might not
ii. Are not
iii. Might be

Concept Check

6. Suppose S is a 5 dimensional subspace of $\mathbb{R}^6$. True or false (example if false):
   i. Every basis for S can be extended to a basis for $\mathbb{R}^6$ by adding one more vector
   ii. Every basis for $\mathbb{R}^6$ can be reduced to a basis for S by removing one vector

Solution:

i. True (HW 1.3)

ii. False (See recitation of Week 1)

Concept Check

7. $(*)$We can think of a permutation of n elements as a linear transformation $P: \mathbb{R}^n \to \mathbb{R}^n$ that permutes the basis elements $e_1, \dots, e_n$. In that case it is an $n \times n$ matrix with only 0's and 1's as entries and such that every row and every column have exactly one entry being 1. Is it true that for any n and any permutation P there exists an integer $k \geq 1$ such that $P^k = I$, where I is the identity matrix. Justify your answer.

Solution:

Recall from class that a linear transformation is uniquely determined by its action on a basis. Thus, as there are n! distinct permutations of the standard basis vectors $e_1, \ldots, e_n$, there are exactly n! distinct permutations matrices. Consider the list:

$$P, P^2, P^3, \dots, P^{n!+1}$$

We first establish that for all i > 0, P^i is a permutation matrix. We do this by proving that the composition of permutations is a permutation. Using the fact that a linear transformation is determined by its action on a basis, we can reduce this to showing that a composition of permutations of the standard basis vectors is still a permutation.

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Since we now know every P^i in our list is a permutation, the pigeonhole principle says there must exist integers 1 ≤ i < j ≤ n! + 1 such that $P^i = P^j$. Equivalently, we can say that $P^i = P^{j-i}P^i$. Thus, for any standard basis vector e_k, we have

$$P^i e_k = P^{j-i} P^i e_k$$

Since P^i is a permutation,

$$\{P^i e_k : k \in [1, n]\} = \{e_k : k \in [1, n]\}$$

Thus $P^{j-i} e_k = e_k$ for k = 1, . . . , n. As P^{j-i} is uniquely determined by its action on a basis, we see that $P^{j-i} = I$ as required.

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