Concept check Solutions - Week 1

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Concept check Questions

1. Describe a subspace S of each vector space V and then a subspace SS of S:
   - $V_1 = all\ combinations\ of\ (1,1,0,0), (1,0,1,0)\ and\ (1,1,1,1)$
   - $V_2 = all\ symmetric\ 2\ by\ 2\ matrices$
   - $V_3 = all\ solutions\ to\ the\ equation \frac{d^4y}{dx^4} = 0$
2. Start with vectors $v_1 = (1,2,0)\ and\ v_2 = (2,3,0)$
   - Are $v_1\ and\ v_2$ linearly independent?
   - Are they a basis for any space?
   - What space V do they span?
   - What is the dimension of V?
   - Describe all vectors v_3 such that v_1, v_2, v_3 completes a basis of $\mathbb{R}^3$
3. Let w_1, w_2, w_3 be independent vectors. What can you say about the independence of $w_1 - w_2$, $w_3 - w_1, w_2 - w_3$? What about $w_1 + w_2, w_1 + w_3, w_2 + w_3$?

Solutions

1. Describe a subspace S of each vector space V and then a subspace SS of S:
   a) $V_1 = \text{all combinations of } (1,1,0,0), (1,0,1,0) \text{ and } (1,1,1,1)$
   b) $V_2 = \text{all symmetric 2 by 2 matrices}$
   c) $V_3 = \text{all solutions to the equation } \frac{d^4y}{dx^4} = 0$

Solution:

a) $S = \text{all combinations of } (1,1,0,0), (1,0,1,0) \text{ and } SS = \text{span of } (1,1,0,0)$

b) $S = \text{all diagonal 2 by 2 matrices and } SS =$
   $\text{all 2 by 2 matrices with a real number at } (0,0) \text{ position and 0 everywhere else}$

c) $S = \text{space of polynomials with degree at most 3}, \ SS =$
   $\text{space of polynomials of degree at most 2}$

Concept check Questions

2. Start with vectors $v_1 = (1,2,0)\ and\ v_2 = (2,3,0)$
   a) Are $v_1\ and\ v_2$ linearly independent?
   b) Are they a basis for any space?
   c) What space V do they span?
   d) What is the dimension of V?
   e) Describe all vectors v_3 such that v_1, v_2, v_3 completes a basis of $\mathbb{R}^3$

Solution:

a) Yes, they are linearly independent. To see why:

$$av_1 + bv_2 = 0$$
$$\Rightarrow (a + 2b, 2a + 3b, 0) = 0$$
$$\Rightarrow a + 2b = 0\ and\ 2a + 3b = 0$$
$$\Rightarrow \mathrm{a} = 0 \text{ and } \mathrm{b} = 0$$

Thus, the linear combination of the vectors=0 only for all zero coefficients proving that they are linearly independent.

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2. Start with vectors $v_1 = (1,2,0)\ and\ v_2 = (2,3,0)$
   a) Are $v_1\ and\ v_2$ linearly independent?
   b) Are they a basis for any space?
   c) What space V do they span?
   d) What is the dimension of V?
   e) Describe all vectors v_3 such that v_1, v_2, v_3 completes a basis of $\mathbb{R}^3$

Solution:

b) They are a basis for subspace $S = span\big((1,2,0),(2,3,0)\big) = (x, y, 0)\ \forall\ x, y \in \mathbb{R}$

c) They span the x-y plane in $\mathbb{R}^3$ i.e $V = (x, y, 0) \forall\ x, y \in \mathbb{R}$

d) Dimension of V=2

e) $v_3 = \{(x, y, k) | x, y, k \in \mathbb{R}\ and\ k \neq 0\}$

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3. Let w_1, w_2, w_3 be independent vectors. What can you say about the independence of $w_1 - w_2$, $w_3 - w_1, w_2 - w_3$? What about $w_1 + w_2, w_1 + w_3, w_2 + w_3$?

Solution:

To check if $w_1 - w_2,\ w_3 - w_1, w_2 - w_3$ are linearly independent or not:

$$a(w_1 - w_2) + b(w_3 - w_1) + c(w_2 - w_3) = 0$$
$$\Rightarrow (a - b)w_1 + w_2(c - a) + w_3(b - c) = 0$$

Since w_1, w_2, w_3 are linearly independent,

$$a - b = 0, c - a = 0, b - c = 0$$
$$\Rightarrow a = b = c$$

We don't need to have all coefficients =0 for the linear combination of $w_1 - w_2,\ w_3 - w_1, w_2 - w_3$ to be zero. Therefore, $w_1 - w_2,\ w_3 - w_1, w_2 - w_3$ are linearly dependent

Concept check Questions

3. Let w_1, w_2, w_3 be independent vectors. What can you say about the independence of $w_1 - w_2$, $w_3 - w_1, w_2 - w_3$? What about $w_1 + w_2, w_1 + w_3, w_2 + w_3$?

Solution:

To check if $w_1 + w_2,\ w_3 + w_1, w_2 + w_3$ are linearly independent or not:

$$a(w_1 + w_2) + b(w_3 + w_1) + c(w_2 + w_3) = 0$$
$$\Rightarrow (a + b)w_1 + w_2(c + a) + w_3(b + c) = 0$$

Since w_1, w_2, w_3 are linearly independent,

$$a + b = 0, c + a = 0, b + c = 0$$
$$\Rightarrow a = b = c = 0$$

All coefficients = 0 if the linear combination of $w_1 + w_2,\ w_3 + w_1, w_2 + w_3$ has to be zero. Therefore, $w_1 + w_2,\ w_3 + w_1, w_2 + w_3$ are linearly independent